BENIN JOURNAL OF
STATISTICS
ISSN 2682-5767
Vol. 3, pp. 12– 40 (2020)

Lambda Upper Bound Distribution: Some Properties and Applications

M. I. Ekum[1*], I. A. Adeleke[2] and E. E. Akarawak[3]

[1,3] *Department of Mathematics, University of Lagos, Akoka, Yaba, Lagos, Nigeria*; [2]*Department of Actuarial Science and Insurance, University of Lagos, Akoka, Yaba, Lagos, Nigeria*

Abstract. In this work, we proposed a new univariate continuous distribution derived from one-parameter beta distribution using the property of probability density function (pdf) and the theory of Riemann integration. The proposed distribution is called Lambda Upper Bound (LUB) Distribution because it is bounded above by Lambda (λ). The proposed distribution with two parameters can also be referred to as two parameter power function distribution and will present new opportunities for assessing reliability and survival data in different field of life such as environmental, engineering, medicine and finance. The distribution was characterized using different functions and different properties of the distribution were derived including the Shannon entropy, order statistics and moment. We used maximum likelihood estimation method to estimate the distribution parameters and they are in closed form. Simulation study was carried out to test the consistency of the parameters estimates and applied two real life data and compared the result with existing distributions. The result of the comparison shows that the proposed distribution, despite having only two parameters fitted well to the two data sets and compared favourably well with other more complex distributions. The proposed distribution is a type of beta distribution with shape and scale parameters and can be used as an alternative to beta, kumaraswamy and uniform distributions.

Keywords: Beta distribution, Kumaraswamy distribution, lambda upper bound function, maximum likelihood estimation, power function.

Published by: Department of Statistics, University of Benin, Nigeria

1. Introduction

Generating new probability distributions, finding their properties and their real life applications are what is trending in the field of mathematical statistics and probability (Famoye et al., 2018). Univariate continuous probability distributions are commonly applied to describe real world phenomena. Due to the usefulness of these distributions, their theory is widely studied and new distributions are developed. Lee et al. (2013) reviewed methods of generating these univariate continuous probability distributions in recent decades. The interest of developing more probability distributions that are flexible remain strong in statistics profession. McDonald (1984); and Bookstaber and McDonald (1987) studied the generalized beta of the first and second kind respectively, which were applied to study the distribution of income and describing stock price returns respectively.

There are many univariate continuous probability distributions that are supported on the interval $[0, \infty)$ but in some real situations random variables assume finite upper bound. For example, students results are bounded on the interval [0, 100], test or continuous assessment scores in some cases are bounded on the interval [0, 30], probability values are bounded on the interval [0, 1], measures of partitions in statistics like percentile [0, 100], decile [0, 10], octile [0, 8], quartile [0, 4] and so on. The Beta distribution is one of the most basic distributions supported on finite range (0, 1). The Beta distribution has been widely studied in both practice and theory in Mathematical Statistics by Nadarajah and Kotz (2004 and 2005), Kong et al. (2007), Akinsete et al. (2008) Pescim et al. (2010), Souza et al. (2010), Nassarand and Elmagry (2012), Nassar and Nada (2011, 2012 and 2013) and Anake et al. (2015). An alternative distribution to the beta distribution, which is easier to work with, is the Kumaraswamy distribution proposed by Kumaraswamy (1980).

The beta distribution as earlier mentioned, in probability theory and statistics in particular, is a family of

*Corresponding author. Email: matekum@yahoo.com

continuous probability distributions defined on the interval [0, 1] with two positive shape parameters, α and β, that control the shape of the distribution. The beta distribution has been applied to model the behavior of random variables limited to intervals of finite length between 0 and 1 (Kieschnick and McCullough, 2003 and Johnson et al., 1995). The Kumaraswamy distribution is much simpler to use when compared with the beta distribution, especially in simulation studies due to the simple closed form of both its probability density function and cumulative distribution function. Another distribution in this class is the uniform distribution bounded on a closed interval $[a, b]$ (Park and Bera, 2009).

Many individual and generalized convoluted classes of these distributions have been developed and applied to describe various real phenomena even in the areas of environmental hazards and engineering. A common feature of these generalized convoluted distributions is that they have more parameters. A major disadvantage of these distributions with more than three parameters is in the estimation of their parameters and making inferences with them, especially with the ones that do not have closed form of their maximum likelihood estimates or moments. Eugene et al. (2002) proposed the beta-generated family of distributions using the beta as the baseline distribution.

In this paper, the beta distribution is used as the baseline distribution. The shape parameters of the beta distribution are redefined to determine the shape and upper bound of the proposed distribution.

The remaining parts of the paper are unfolded as follows. Section 2 presents the proposed distribution and its function. Section 3 presents the characterized proposed distribution by its probability density function (pdf), cumulative distribution function (cdf), survival function, hazard function, cumulative hazard function and quantile function. Also, the limiting function, Shannon entropy, order statistics, and related distributions are presented. In section 4, we discuss the moments and maximum likelihood estimation (mle) of the parameters.Section 5 presents the distribution fitting, while section 6 presents the simulation study and shows the consistency of the distribution parameters. Applications based on real data are presented in Section 7. Finally, concluding remarks are given in Section 8.

2. Derivation of Lambda Upper Bound (LUB) distribution

The beta distribution is a flexible and useful tool in modelling continuous random variables that assume values in the standard unit interval (0, 1), such as rates, percentages and proportions (Kieschnick and McCullough, 2003). In particular, one may be interested in modelling fluctuations in variables such as income concentration, unemployment rate, and proportion of votes for an incumbent president running for reelection, whose support is the standard unit interval. Beta distribution is bounded on the interval [0, 1], which is measured in standard unit, but we are not only interested in values measured in standard units but also values that are greater than 1. In this case, one may be interested in modelling fluctuations in variables such as income, expenditure and quantity of olive oil in a 25cl bottle, whose support are not in the standard unit interval.

PROPOSITION 2.1 *Let X be a random variable that follows a beta distribution with parameters α and β, such that $\alpha = a$ and $b > 0$, then by integrating the resulting probability density function (pdf) of X, $u(x)$ on the closed interval $[0, \lambda]$ then the pdf $f(x)$ is given by*

$$f(x) = \{^{\frac{a}{\lambda^a} x^{a-1},\, 0 \le x \le \lambda,\, a>0,\, \lambda>0}_{0,\, otherwise}$$

Proof. The pdf, $g(x)$ of beta distribution with parameters α and β, that is , $X \sim Be(\alpha, \beta)$ is given by

$$g(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)},\ x \ge 0,\ \alpha, \beta > 0, \tag{1}$$

where α and β are the shape parameters and the values of X are between 0 and 1. By letting $\alpha = a$ and $\beta = 1$, that is $X \sim Be(a, 1)$, will reduce (1) to another pdf, $u(x)$.

$$u(x) = x^{a-1},\ x \ge 0,\ \alpha > 0. \tag{2}$$

Note that (2) is another proper pdf that follows a beta distribution with parameters $\alpha = a$ and $\beta = 1$. But we are interested in a flexible upper bound which is a positive real number, which might be different from 1, say

λ. It is expected that if the support $[0, \lambda]$ specifies the total probability space, such that the definite integral bounded in the closed interval $[0, \lambda]$, where $\lambda > 0$ is given by

$$\int_0^\lambda u(x)dx, \tag{3}$$

then (3) will be equated to 1, since it defines the total probability space

$$\int_0^\lambda x^{a-1}dx = 1, \tag{4}$$

Rearrange equation (4) to have

$$\frac{a}{\lambda^a}\int_0^\lambda x^{a-1}dx = 1, \tag{5}$$

and by one of the properties of a true pdf we have

$$\int_0^\lambda \frac{a}{\lambda^a}x^{a-1}dx = \int_0^\lambda f(x)dx = 1. \tag{6}$$

Thus,

$$f(x) = \{^{\frac{a}{\lambda^a}x^{a-1}, 0\leq x\leq\lambda, a>0, \lambda>0}_{0, otherwise} \tag{7}$$

■

So, (7) is the proposed Lambda Upper Bound (LUB) distribution provided the values of x are within the support $[0, \lambda]$, where a and λ are the shape and scale parameters respectively. Therefore, the LUB function is given by

$$\int_0^\lambda x^{a-1}dx = \frac{\lambda^a}{a}. \tag{8}$$

The LUB function can be used to evaluate integral of the form in (8), for example.

$$\int_0^5 x^3 dx = 156.25 \tag{9}$$

The solution of the definite integral in (9) can be derived without doing actual integration by comparing the integral with LUB function, we have: $\lambda = 5$, $a - 1 = 3$, so that $a = 4$. Therefore,

$$\frac{5^4}{4} = 156.25$$

3. Characterization of the LUB distribution

3.1 *Probability density function*

One of the ways of characterizing a random variable is through its probability density function (pdf). The LUB distribution defined in (8) is a special case of the beta distribution, which was proved by Proposition 2.1.

3.2 *Cumulative distribution function*

THEOREM 3.1 *Let X be a LUB distributed random variable. The cumulative distribution function (cdf), $F(x)$ of LUB distribution is a power function of the cdf $V(x)$ of a uniform distribution defined on a closed interval* $[0, \lambda]$.

Proof. By definition, the cumulative distribution function (cdf) of a random variable X is given by

$$F(x) = \int_0^x f(t)dt \tag{10}$$

Given the LUB distribution in (7), we obtain the cdf of X as

$$F(x) = \int_0^x \frac{at^{a-1}}{\lambda^a}dt = \left[\frac{t^a}{\lambda^a}\right]_0^x. \tag{11}$$

Solving (11) further gives (12)

$$F(x) = \frac{x^a}{\lambda^a} = \left(\frac{x}{\lambda}\right)^a, \tag{12}$$

which can also be written as

$$P(X \leq x) = F(x) = [V(x)]^a.$$

where $V(x) = \frac{x}{\lambda}$ is the cdf of a uniform distribution defined on a closed interval $[0, \lambda]$. ■

Thus, the cdf of LUB distribution is the cdf of uniform distribution raised to the power of a. This implies that if parameter $a = 1$, the the cdf of LUB distribution will be reduced to that of uniform distribution.

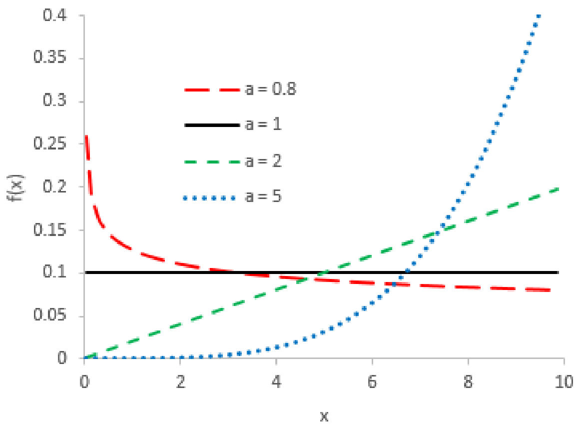


Figure 1: The pdf of LUB distribution for different values of a and $\lambda = 10$

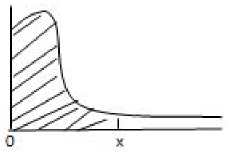
Fig. 1 depicts the pdf of LUB distribution for different values of parameter a for a constant value of parameter λ. It shows that for a constant λ, if $a < 1$, it is skewed to the right and if $a > 1$, it is skewed to the left, but if $a = 1$, it is symmetric. The values of $P(X \leq x) = p$ for some values of x in the closed interval $[0, \lambda = 100]$ and for different values of a is shown in Table 1. Fig. 2. depicts the cdf of LUB distribution for various values of a at constant $\lambda = 10$. It shows that the values of the cdf increased from 0 to 1 as the value of x increased from 0 to 10.

3.3 *Survival function*

By definition, the survival function $S(x)$ of a random variable X, is a function that gives the probability that an item of interest will survive beyond any given specified time, also known as the reliability function and is

Table 1: The values of $P(X \leq x) = p$ at different values of a at constant $\lambda = 100$

$F(X \leq x) = p$



	a												
X	0.001	0.05	0.075	0.1	0.5	1	2	2.5	5	10	25	50	100
0.5	0.9947	0.7673	0.6721	0.5887	0.0707	0.0050	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.9954	0.7943	0.7079	0.6310	0.1000	0.0100	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.9961	0.8223	0.7457	0.6762	0.1414	0.0200	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.9965	0.8392	0.7687	0.7042	0.1732	0.0300	0.0009	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.9968	0.8513	0.7855	0.7248	0.2000	0.0400	0.0016	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.9970	0.8609	0.7988	0.7411	0.2236	0.0500	0.0025	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9972	0.8688	0.8098	0.7548	0.2449	0.0600	0.0036	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.9973	0.8755	0.8192	0.7665	0.2646	0.0700	0.0049	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9975	0.8814	0.8274	0.7768	0.2828	0.0800	0.0064	0.0018	0.0000	0.0000	0.0000	0.0000	0.0000
9	0.9976	0.8866	0.8348	0.7860	0.3000	0.0900	0.0081	0.0024	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9977	0.8913	0.8414	0.7943	0.3162	0.1000	0.0100	0.0032	0.0000	0.0000	0.0000	0.0000	0.0000
15	0.9981	0.9095	0.8674	0.8272	0.3873	0.1500	0.0225	0.0087	0.0001	0.0000	0.0000	0.0000	0.0000
20	0.9984	0.9227	0.8863	0.8513	0.4472	0.2000	0.0400	0.0179	0.0003	0.0000	0.0000	0.0000	0.0000
25	0.9986	0.9330	0.9013	0.8706	0.5000	0.2500	0.0625	0.0313	0.0010	0.0000	0.0000	0.0000	0.0000
30	0.9988	0.9416	0.9137	0.8866	0.5477	0.3000	0.0900	0.0493	0.0024	0.0000	0.0000	0.0000	0.0000
35	0.9990	0.9489	0.9243	0.9003	0.5916	0.3500	0.1225	0.0725	0.0053	0.0000	0.0000	0.0000	0.0000
40	0.9991	0.9552	0.9336	0.9124	0.6325	0.4000	0.1600	0.1012	0.0102	0.0001	0.0000	0.0000	0.0000
45	0.9992	0.9609	0.9419	0.9233	0.6708	0.4500	0.2025	0.1358	0.0185	0.0003	0.0000	0.0000	0.0000
50	0.9993	0.9659	0.9493	0.9330	0.7071	0.5000	0.2500	0.1768	0.0313	0.0010	0.0000	0.0000	0.0000
55	0.9994	0.9706	0.9562	0.9420	0.7416	0.5500	0.3025	0.2243	0.0503	0.0025	0.0000	0.0000	0.0000
60	0.9995	0.9748	0.9624	0.9502	0.7746	0.6000	0.3600	0.2789	0.0778	0.0060	0.0000	0.0000	0.0000
65	0.9996	0.9787	0.9682	0.9578	0.8062	0.6500	0.4225	0.3406	0.1160	0.0135	0.0000	0.0000	0.0000
70	0.9996	0.9823	0.9736	0.9650	0.8367	0.7000	0.4900	0.4100	0.1681	0.0282	0.0001	0.0000	0.0000
75	0.9997	0.9857	0.9787	0.9716	0.8660	0.7500	0.5625	0.4871	0.2373	0.0563	0.0008	0.0000	0.0000
80	0.9998	0.9889	0.9834	0.9779	0.8944	0.8000	0.6400	0.5724	0.3277	0.1074	0.0038	0.0000	0.0000
85	0.9998	0.9919	0.9879	0.9839	0.9220	0.8500	0.7225	0.6661	0.4437	0.1969	0.0172	0.0003	0.0000
90	0.9999	0.9947	0.9921	0.9895	0.9487	0.9000	0.8100	0.7684	0.5905	0.3487	0.0718	0.0052	0.0000
95	0.9999	0.9974	0.9962	0.9949	0.9747	0.9500	0.9025	0.8796	0.7738	0.5987	0.2774	0.0769	0.0059
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

given by

$$S(x) = 1 - F(x) \tag{13}$$

THEOREM 3.2 *The survival function of a LUB distribution $S(x)$ with cdf $F(x)$ is a power function of the survival function $S_V(x)$ of a uniform distribution with cdf $V(x)$ defined on a closed interval* $[0, \lambda]$.

$$S(x) = (S_V(x))^a \tag{14}$$

Proof. Given the cdf in (12), we derive the survival function by

$$S(x) = 1 - \frac{x^a}{\lambda^a} \tag{15}$$

$$S(x) = \frac{\lambda^a - x^a}{\lambda^a} = \left(\frac{\lambda - x}{\lambda}\right)^a \tag{16}$$

where $\frac{\lambda - x}{\lambda}$ is the survival function $S_V(x)$ of a uniform distribution defined on the interval $[0, \lambda]$. ■

Fig. 3 depicts the survival function of LUB distribution. It shows that the values of the survival function decreased from 1 to 0 as the value of x increased from 0 to 10.

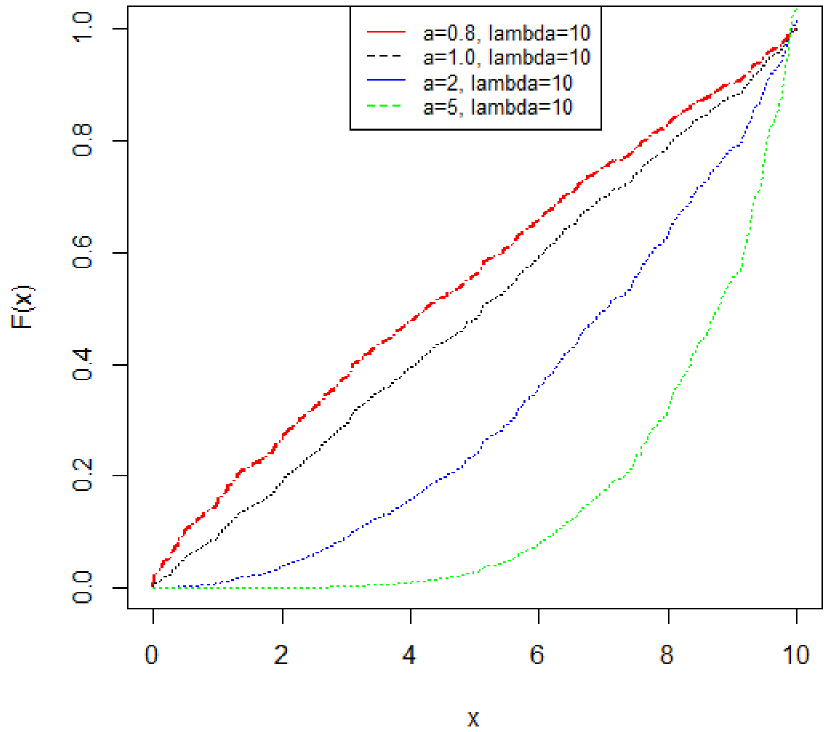


Figure 2: The cdf of LUB distribution for different values of a

3.4 *Hazard function*

By definition, the hazard function $h(x)$ of a random variable X also known as the failure rate or force of mortality is the ratio of $f(x)$ to $S(x)$, and is given by

$$h(x) = \frac{f(x)}{S(x)} \tag{17}$$

PROPOSITION 3.3 *The hazard function of a random variable X that follows a LUB distribution with parameters a = 1 and λ will reduce to the hazard function of a uniform distributed on the interval [0, λ].*

Proof. Given the pdf in (7) and the survival rate in (16), we derive the h(x) of LUB distribution as

$$h(x) = \frac{ax^{a-1}}{\lambda^a - x^a}. \tag{18}$$

Put a = 1, we have

$$h(x) = \frac{1}{\lambda - x}, \tag{19}$$

which is the hazard function of uniform distribution defined on the interval $[0, \lambda]$. ■

3.5 *Cumulative hazard function*

THEOREM 3.4 *Let X be random variable that follows a LUB distribution with cdf, $F(x)$. The cumulative hazard function, $H(x)$ of LUB distribution is the product of parameter a and the cumulative hazard function, $H_V(x)$ of uniform distribution with cdf $V(x)$ defined on the interval $(0, \lambda)$.*

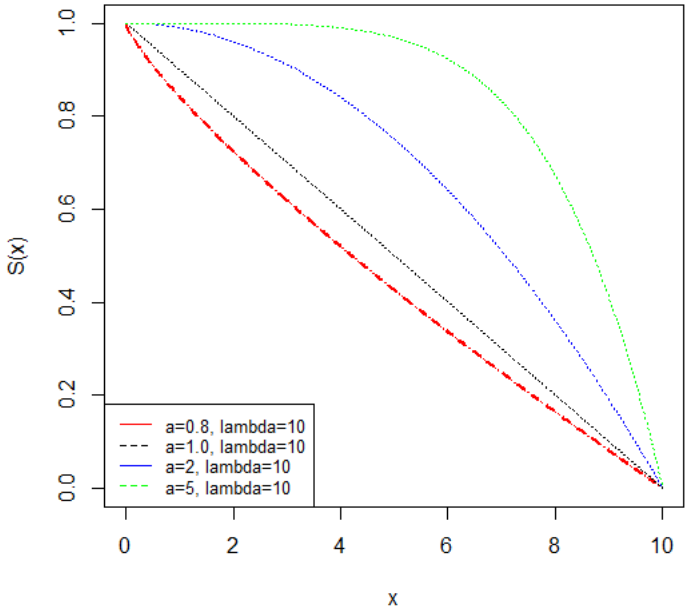


Figure 3: The Survival Function of LUB distribution for different values of a

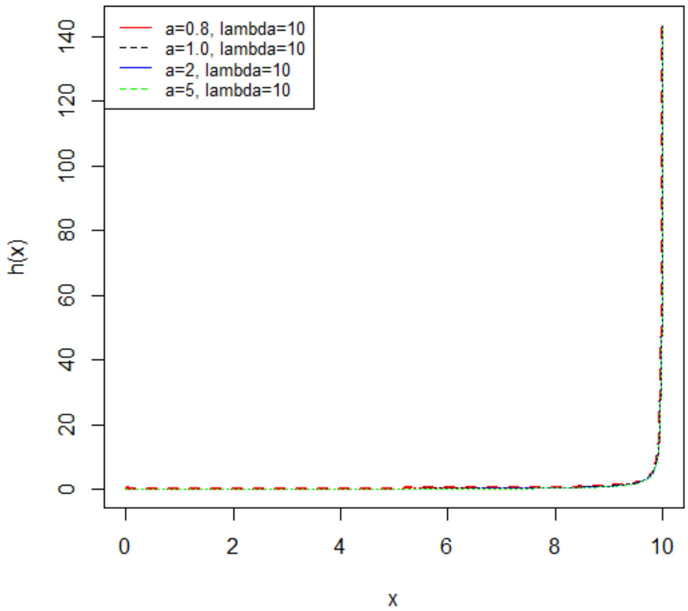


Figure 4: The Hazard Function of LUB Distribution for different values of a

Proof. By definition, the cumulative hazard function $H(x)$ of a random variable X is given by

$$H(x) = -lnS(x) \tag{20}$$

Given the survival function (16), we derive the $H(x)$ of LUB distribution as

$$H(x) = -ln\left(\frac{\lambda - x}{\lambda}\right)^a. \tag{21}$$

Equation (21) can be written as (22)

$$H(x) = -aln\left(\frac{\lambda - x}{\lambda}\right) = aH_V(x), \tag{22}$$

Equation (22) is the cumulative hazard function of LUB distribution where

$$H_V(x) = -ln\left(\frac{\lambda - x}{\lambda}\right). \tag{23}$$

is the cumulative hazard function of a uniform distribution defined on the interval $(0, \lambda)$. ■

The cumulative hazard function of LUB distribution at different values of a and constant λ is depicted in Fig. 5. Fig. 4 and Fig. 5 depict the hazard and the cumulative hazard rates of LUB distribution. Fig. 4 and

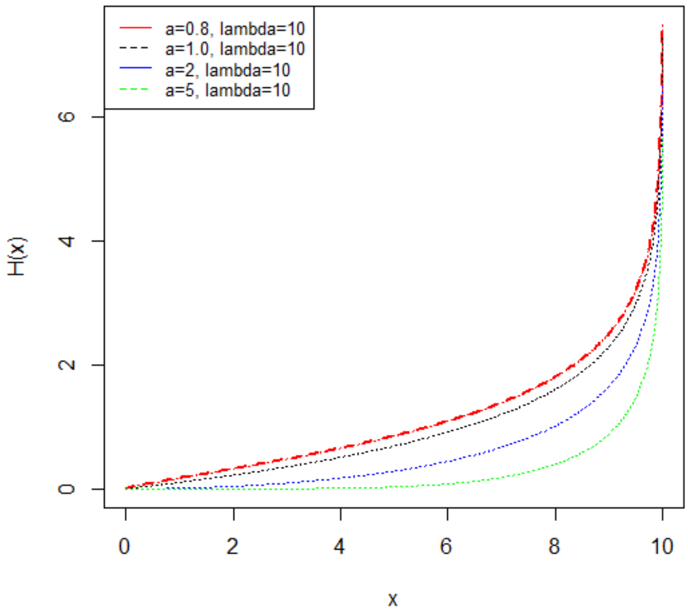


Figure 5: The Cumulative Hazard Function of LUB Distribution for different values of a

Fig. show that both the hazard rate and cumulative hazard rate are asymptotic to both horizontal and vertical axes.

3.6 *Quantile function*

The quantile function of a random variable X is the value at which the probability of the random variable is less than or equal to the given probability. It is the inverse function of the cdf and it is defined as

$$Q(x) = F(x^{-1}) \tag{24}$$

Given the cdf in (12), we derive $Q(x)$ by making x the subject of the formula

$$x = (\lambda^a F(x))^{1/a}.$$

Therefore

$$Q(x) = F(x^{-1}) = (\lambda^a F(x))^{1/a}$$

Table 2: The x values at difference values of $P(X \leq x)$ for different values of $a > 0$, for constant $\lambda = 100$

	$P(X \leq x) = p$										
a	0.005	0.01	0.02	0.025	0.05	0.1	0.5	0.9	0.975	0.99	1
0.001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0043	100.0000
0.02	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5154	28.1988	60.5006	100.0000
0.05	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	12.1577	60.2688	81.7907	100.0000
0.075	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0097	24.5414	71.3501	87.4586	100.0000
0.1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0977	34.8678	77.6330	90.4382	100.0000
0.5	0.0025	0.0100	0.0400	0.0625	0.2500	1.0000	25.0000	81.0000	95.0625	98.0100	100.0000
1	0.5000	1.0000	2.0000	2.5000	5.0000	10.0000	50.0000	90.0000	97.5000	99.0000	100.0000
2	7.0711	10.0000	14.1421	15.8114	22.3607	31.6228	70.7107	94.8683	98.7421	99.4987	100.0000
2.5	12.0112	15.8489	20.9128	22.8653	30.1709	39.8107	75.7858	95.8732	98.9924	99.5988	100.0000
5	34.6572	39.8107	45.7305	47.8176	54.9280	63.0957	87.0551	97.9148	99.4949	99.7992	100.0000
10	58.8704	63.0957	67.6243	69.1503	74.1134	79.4328	93.3033	98.9519	99.7471	99.8995	100.0000
25	80.9019	83.1764	85.5148	86.2815	88.7072	91.2011	97.2655	99.5794	99.8988	99.9598	100.0000
50	89.9455	91.2011	92.4742	92.8878	94.1845	95.4993	98.6233	99.7895	99.9494	99.9799	100.0000
100	94.8396	95.4993	96.1635	96.3783	97.0487	97.7237	99.3092	99.8947	99.9747	99.9900	100.0000

The quantile function returns the value x such that

$$F(x) = P(X \leq x) = p$$

So, that the quantile function of LUB distribution can be written as

$$x = Q(p) = \lambda p^{1/a} \tag{25}$$

The quantile function of a particular distribution is used in Monte Carlo method to simulate random variates that follows such distribution.

Table 2 displays the x values at difference values of $p = P(X \leq x)$ for different values of $a > 0$, for constant $\lambda = 100$. It can be seen that the values of x ranges from 0 to 100. The higher the p, the higher the value of x for a particular a.

The histograms of LUB distribution for different values of a at constant λ is depicted in Fig. 6. It shows that for a constant λ, if $a < 1$, it is skewed to the right and if $a > 1$, it is skewed to the left, but if $a = 1$, it is symmetric. The shape of LUB distribution is solely determined by the value of a. So, LUB distribution can be positively skewed, negatively skewed or symmetric, depending on the value of the shape parameter, a. R code used in generating the plots are shown in Appendix I. Some of the plots and tables are generated using Microsoft Excel formulas.

3.6.1 Median

The median of a probability distribution is the value separating the higher half the probability distribution from its lower half. It is the ”middle” value.

Given the quantile function in (25), the median of Lambda upper bound distribution is given by

$$Q(0.5) = \lambda \left(\frac{1}{2}\right)^{1/a} \tag{26}$$

3.6.2 Measures of partition

The quantile function can be used to partition a distribution into different non-overlapping continuous sections. We can determined the quartiles, octiles, deciles and percentiles using the quantile function.

The first and the third quartiles are given by (27) and (28) respectively

$$Q(0.25) = \lambda \left(\frac{1}{4}\right)^{1/a} \tag{27}$$

$$Q(0.75) = \lambda \left(\frac{3}{4}\right)^{1/a}. \tag{28}$$

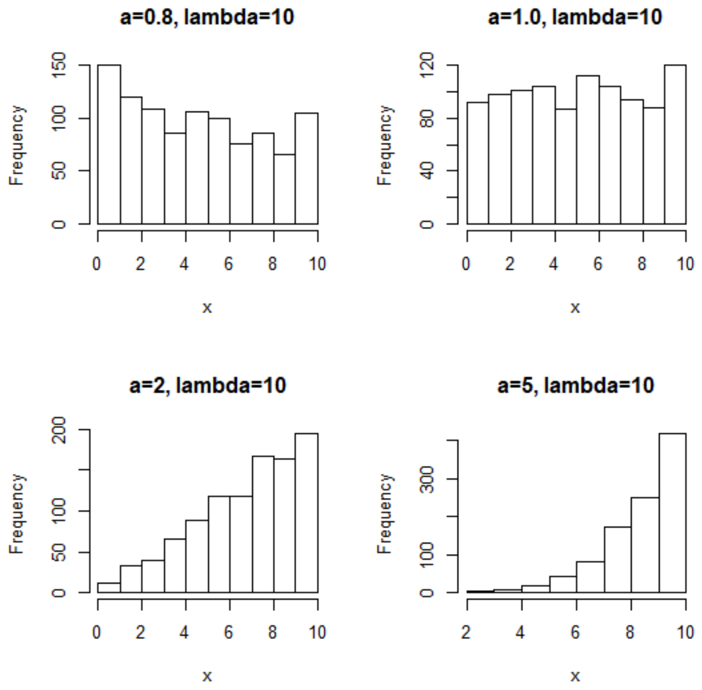


Figure 6: The histogram of LUB distribution for different values of a

3.7 *Mode of LUB distribution*

The mode of LUB distribution is parameter λ. According to Johnson et al. (1995), the mode of a beta distributed random variable X with parameters $\alpha, \beta > 1$, defined on the interval $[0, 1]$ corresponding to the peak in the pdf is given by

$$Mode = \frac{\alpha - 1}{\alpha + \beta - 2} \tag{29}$$

So, letting $\beta = 1$, the Mode in (29) is equal to 1, corresponding to the peak in the pdf given in (1). Thus, the peak of LUB distribution corresponding to the peak in the pdf given in (8) is $lambda$.

3.8 *Limiting functions of LUB distribution*

1a. The limit of $f(x)$ in (8) as $x \to 0$ is equal to zero

$$\lim_{x \to 0} f(x) = 0 \tag{30}$$

$$\lim_{x \to 0} \frac{a}{\lambda^a} x^{a-1} = 0 \tag{31}$$

1b. The limit of $f(x)$ in (8) as $x \to \lambda$ is equal to $\frac{a}{\lambda}$

$$\lim_{x\to\lambda} f(x) = \frac{a}{\lambda} \tag{32}$$

$$\lim_{x\to\lambda} \frac{a}{\lambda^a} x^{a-1} = \frac{a}{\lambda} \tag{33}$$

2a. The limit of $F(x)$ in (12) as $x \to 0$ is equal to 0

$$\lim_{x\to 0} F(x) = 0 \tag{34}$$

$$\lim_{x\to 0} \frac{x^a}{\lambda^a} = 0 \tag{35}$$

2b. The limit of $F(x)$ in (12) as $x \to \lambda$ is equal to 1

$$\lim_{x\to\lambda} F(x) = 1 \tag{36}$$

$$\lim_{x\to\lambda} \frac{x^a}{\lambda^a} = 1 \tag{37}$$

3.9 *Shannon entropy*

Shannon entropy of a random variable X is a measure of variation of uncertainty and it is defined by

$$E[-log f(x)]. \tag{38}$$

Using the LUB distribution, equation (38) gives (39)

$$E(-log f(x)) = E\left[-log\left(\frac{a}{\lambda^a} x^{a-1}\right)\right]. \tag{39}$$

Thus, the Shannon entropy of LUB distribution becomes (40)

$$\Phi(x) = -loga + alog\lambda - (a-1)E(logx) \tag{40}$$

where $\Phi(x)$ is the shannon entropy of Lamda upper bound distribution

3.10 *Order statistics*

3.10.1 1^{st} *Order statistics*

Let $X_1, X_2, ..., X_n$ be a random sample from the LUB distribution and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, such that, $X_{(1)} \leq X_{(2)} \leq \ldots\ldots \leq X_{(n)}$, are order statistics obtained from the sample. Then the pdf $f_{X_1}(x)$ of the 1^{st} order statistics, $X_{(1)}$ is given by

$$f_{X_1}(x) = -\frac{d}{dx} \prod_{i=1}^{n} [1 - F(x)]^n = n\,[1 - F(x)]^{n-1} f(x) \tag{41}$$

but $S(x) = 1 - F(x)$, so that (49) becomes

$$f_{X_1}(x) = n\,[S(x)]^{n-1} f(x) \tag{42}$$

Substitute (7) and (16) into (42) to obtain

$$f_{X_1}(x) = n\left[1 - \frac{x^a}{\lambda^a}\right]^{n-1}\frac{a}{\lambda^a}x^{a-1} \tag{43}$$

Thus, (44) is the 1st order statistics of LUB distribution.

$$f_{X_1}(x) = \frac{an}{\lambda^{an}}x^{a-1}\left[\lambda^a - x^a\right]^{n-1} \tag{44}$$

3.10.2 n^{th} Order statistics

THEOREM 3.5 *The pdf $f_{X_n}(x)$ of the n^{th} order statistics of LUB distribution is the same as the pdf of LUB distribution with shape parameter, an and scale parameter, λ, where an is the sum of parameter a in n places.*

Proof. Let $X_1, X_2, ..., X_n$ be a random sample from the LUB distribution and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, such that, $X_{(1)} \leq X_{(2)} \leq \ldots\ldots \leq X_{(n)}$, are order statistics obtained from the sample. Then the pdf $f_{X_n}(x)$ of the n^{th} order statistics, $X_{(n)}$ is given by

$$f_{X_n}(x) = n\left[F(x)\right]^{n-1}f(x) \tag{45}$$

Substitute (8) and (12) into (53), we have (54)

$$f_{X_n}(x) = n\left[\frac{x^a}{\lambda^a}\right]^{n-1}\frac{a}{\lambda^a}x^{a-1} \tag{46}$$

Thus, (55) is the nth order statistics of LUB distribution which is also a true pdf of LUB distribution with parameters an and λ.

$$f_{X_n}(x) = \frac{an}{\lambda^{an}}x^{an-1} \tag{47}$$

■

3.10.3 General order statistics

THEOREM 3.6 *The pdf of the general order statistics of LUB distribution $f_{X_{(j)}}(x)$ exists and it is given by the expression.*

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\frac{ax^{aj-1}(\lambda^a - x^a)^{n-j}}{\lambda^{an}}$$

Proof. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denote the order statistics of a random sample that follows LUB distribution, $X_1, X_2, ..., X_n$, from a continuous population with cdf, $F(x)$ and pdf $f(x)$. Then the pdf of $X_{(j)}$ is

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}f(x)[F(x)]^{j-1}[1 - F(x)]^{n-j} \tag{48}$$

Substitute (7) and (12) into (48) to have

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\frac{a}{\lambda^a}x^{a-1}\left[\frac{x^a}{\lambda^a}\right]^{j-1}\left[1 - \frac{x^a}{\lambda^a}\right]^{n-j} \tag{49}$$

Thus, the general order statistics of LUB distribution exists and it is given as.

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} \frac{ax^{aj-1}(\lambda^a - x^a)^{n-j}}{\lambda^{an}} \tag{50}$$

■

3.11 *Related distributions*

3.11.1 Uniform distribution

LEMMA 3.7 *Let a random variable X follows a LUB distribution with pdf as defined in (8). If parameter $a = 1$, then f(x) becomes the pdf of a uniform distribution bounded on the closed interval [0, λ] and has the same mean as the LUB distribution for $a = 1$.*

Proof. Recall the pdf of LUB distribution in (8), given by

$$f(x) = \frac{a}{\lambda^a} x^{a-1}, 0 \leq x \leq \lambda, a > 0$$

Let $a = 1$, we have

$$f(x) = \frac{1}{\lambda}, 0 \leq x \leq \lambda$$

By definition, the mean of a random variable X is given by

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

So that we have

$$E(x) = \int_0^\lambda x \frac{1}{\lambda} dx = \frac{1}{\lambda} \int_0^\lambda x dx \tag{51}$$

$$E(x) = \frac{1}{\lambda} \frac{x^2}{2} |_0^\lambda. \tag{52}$$

From (60), we have

$$E(x) = \frac{\lambda}{2} \tag{53}$$

■

Thus, equation (53) is the mean of Uniform distribution bounded on the closed interval $[0, \lambda]$ and (61) is also the mean of LUB distribution for $a = 1$ as we shall see in the estimation of moment in subsequent paragraphs.

3.11.2 Beta distribution

LEMMA 3.8 *Let a random variable X follows a LUB distribution with pdf as defined in (8). If parameter $\lambda = 1$, then f(x) becomes the pdf of a Beta distribution with parameters $\alpha = a$ and $\beta = 1$.*

Proof. Recall the pdf of LUB distribution in (8), given by

$$f(x) = \frac{a}{\lambda^a} x^{a-1}, 0 \leq x \leq \lambda, a > 0$$

Let $\lambda = 1$, we have

$$f(x) = ax^{a-1}, 0 \leq x \leq 1, a > 0 \quad (54)$$

∎

Thus, equation (54) is the pdf of Beta distribution with parameters $\alpha = a$ and $\beta = 1$.

3.11.3 Kumaraswamy distribution

LEMMA 3.9 *Let a random variable X follows a LUB distribution with pdf as defined in (7). If parameter $\lambda = 1$, then $f(x)$ becomes the pdf of a Kumaraswamy distribution with parameter $b = 1$. It can also be referred as a one parameter power function distribution.*

Proof. Recall the pdf of LUB distribution given by

$$f(x) = \frac{a}{\lambda^a} x^{a-1}, 0 \leq x \leq \lambda, a > 0$$

Let $\lambda = 1$, we have

$$f(x) = ax^{a-1}, 0 \leq x \leq 1, a > 0 \quad (55)$$

∎

Thus, equation (55) is the pdf of Kumaraswamy distribution with parameter $b = 1$.

THEOREM 3.10 *Let random variable X follows a LUB distribution with parameters a and $\lambda = 1$, let another random variable Y follows a beta distributions with parameters $\alpha = a$ and $\beta = 1$, and let another random variable Z follows a Kumaraswamy distribution with parameters a and $b = 1$, the $f(x) = f(y) = f(z)$, where $f(x)$, $f(y)$ and $f(z)$ are the pdf of LUB, beta, and Kumaraswamy distributions respectively.*

Proof. Let random variable X follows a LUB distribution with pdf given by

$$f(x) = \frac{a}{\lambda^a} x^{a-1}, 0 \leq x \leq \lambda, a > 0$$

Let $\lambda = 1$, we have

$$f(x) = ax^{a-1}, 0 \leq x \leq 1, a > 0 \quad (56)$$

Let another random variable Y follows a beta distribution with pdf given by

$$f(y) = \frac{1}{B(\alpha, \beta)} y^{\alpha-1}(1-y)^{\beta-1}, 0 \leq y \leq 1, \alpha > 0, \beta > 0 \quad (57)$$

If $\alpha = a$ and $\beta = 1$, then we have

$$f(y) = \frac{1}{B(a, 1)} y^{a-1}(1-y)^{0} = ay^{a-1}, 0 \leq y \leq 1, a > 0 \quad (58)$$

Let another random variable Z follows a Kumaraswamy distribution with pdf given by

$$f(z) = abz^{a-1}(1-z^a)^{b-1}, 0 \leq z \leq 1, a > 0, b > 0 \quad (59)$$

If $b = 1$, then we have

$$f(z) = az^{a-1}, 0 \leq z \leq 1, a > 0 \quad (60)$$

So, from equations (58), (59) and (60), the proof is complete, and we can see that $f(x) = f(y) = f(z)$. ■

4. Estimation

4.1 *Moments*

By definition, the kth moment about the origin of a random variable X is given by

$$E(X^k) = \int x^k f(x)dx \tag{61}$$

Recall the pdf of X in (7) and substitute it in (61) to have

$$E(X^k) = \int_0^\lambda x^k \frac{a}{\lambda^a} x^{a-1} dx = \frac{a}{\lambda^a} \int_0^\lambda x^{k+a-1} dx = \frac{a}{\lambda^a} \left[\frac{x^{k+a}}{k+a} \right]_0^\lambda \tag{62}$$

Thus, the kth moment of LUB distribution about the origin is given by

$$E(X^k) = \frac{a\lambda^k}{k+a} \tag{63}$$

Mean

From (63), if we put $k = 1$, we have the mean of LUB distribution given by

$$E(X) = \frac{a\lambda}{a+1} \tag{64}$$

Variance

By definition, the variance of a random variable X is given by

$$Var(X) = E(X^2) - [E(X)]^2 \tag{65}$$

But

$$E(X^2) = \frac{a\lambda^2}{a+2} \tag{66}$$

Therefore, the variance of LUB distribution is derived by substituting (72) and (74) into (73) to have

$$Var(X) = \frac{a\lambda^2}{a+2} - \left[\frac{a\lambda}{a+1} \right]^2 \tag{67}$$

Solving (67) further, we have

$$Var(X) = \frac{\lambda^2(3a+2)}{(a+1)^2(a+2)} \tag{68}$$

Standard deviation

By definition, the standard deviation of random variable X is the square root of the variance of X and it is given by

$$Sd(X) = \frac{\lambda}{a+1}\sqrt{\frac{3a+2}{a+2}} \tag{69}$$

Coefficient of Variation

THEOREM 4.1 *The coefficient of variation of a random variable X that follows LUB distribution is independent of parameter λ. That is, it does not depend on the range of the data, it only depends on the shape parameter a.*

$$CV(X) = \frac{1}{a}\sqrt{\frac{3a+2}{a+2}} \times 100$$

Proof. By definition, the coefficient of variation of a random variable X is given by

$$CV(X) = \frac{Sd(X)}{E(X)} \times 100 \tag{70}$$

Substitute (68) and (69) in (70) to have

$$CV(X) = \left(\frac{\lambda}{a+1}\sqrt{\frac{3a+2}{a+2}} \div \frac{a\lambda}{a+1}\right) \times 100 \tag{71}$$

Thus, solving (71) further, we have

$$CV(X) = \frac{1}{a}\sqrt{\frac{3a+2}{a+2}} \times 100 \tag{72}$$

Equation (72) concludes the proof that $CV(X)$ is independent of parameter λ. ■

Measure of skewness

PROPOSITION 4.2 *The LUB distribution is symmetric if the shape parameter $a = 1$, irrespective of the value of the scale parameter λ.*

Proof. A distribution is symmetric if its coefficient of skewness is equal to zero, i.e $CS(X) = 0$. By definition, the Pearson Coefficient of Skewness (CS), with mean, $E(X)$, median, (M_d) and standard deviation ($Sd(X)$) is given by

$$CS(X) = 3\left(\frac{E(X) - M_d}{Sd(X)}\right) \tag{73}$$

Substitute (64), (26) and (69)into (73) to have

$$CS(X) = 3\left(\frac{\frac{a\lambda}{a+1} - \lambda\left(\frac{1}{2}\right)^a}{\frac{\lambda}{a+1}\sqrt{\frac{3a+2}{a+2}}}\right) \tag{74}$$

From (74), if $a = 1$, we have

$$CS(X) = 3\left(\frac{\frac{\lambda}{2} - \frac{\lambda}{2}}{\frac{\lambda}{2}\sqrt{\frac{5}{3}}}\right) = 0 \tag{75}$$

Equation (75) concludes the proof and shows that the coefficient of skewness is zero (0), implying that the distribution is symmetric. ■

The skewness of LUB distribution depends so much on the value of a. If $a < 1$, then the distribution is positively skewed. If $a > 1$, then the distribution is negatively skewed, but if $a = 1$, then the distribution is symmetric.

Measure of kurtosis

In probability theory and statistics, kurtosis is a measure of the ”tailedness” of the probability distribution of a random variable X. Kurtosis is a descriptor of the shape of a probability distribution (Westfall, 2014). The kurtosis of any univariate normal distribution is 3. It is common to compare the kurtosis of a distribution to this value. By definition, the Coefficient of Kurtosis (CK), with fourth moment, μ_4 and Variance $(Var(X))$ is given by

$$CK(X) = \frac{\mu_4}{[Var(X)]^2}. \tag{76}$$

But

$$\mu_4 = E[X-\mu]^4 = \int_0^\lambda [x-\mu]^4 f(x)dx \tag{77}$$

where $\mu = \frac{a\lambda}{a+1}$ and $f(x)$ is defined in (7), so that (76) becomes (77)

$$\mu_4 = E[X-\mu]^4 = \frac{a}{\lambda^a}\int_0^\lambda \left[x - \frac{a\lambda}{a+1}\right]^4 x^{a-1}dx \tag{78}$$

Substitute (67) and (78) into (76)

$$CK(X) = \frac{a(a+1)^3(a+2)^2}{\lambda^4(3a+2)^2}\int_0^\lambda [x(a+1) - a\lambda]^4 x^{a-1}dx \tag{79}$$

4.2 *Maximum likelihood estimation*

The likelihood of LUBD is given by

$$Lf(x,a) = \left(\frac{a}{\lambda^a}\right)^n \prod_{i=1}^n x^{a-1} \tag{80}$$

Take the log of (80) to have

$$\ell = ln(Lf(x)) = nlog(a) - anlog(\lambda) + (a-1)\sum_{i=1}^n logx_i \tag{81}$$

Take the derivative of (81) with respect to a gives

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} - nlog(\lambda) + \sum_{i=1}^{n} log(x_i) \tag{82}$$

Equate (82) to zero and solve for a gives

$$\hat{a} = \frac{n}{nlog\lambda - \sum_{i=1}^{n} logx_i} \tag{83}$$

where $\hat{a}$ is the MLE of a and λ is estimated by

$$\hat{\lambda} = max(x_i) + Se(x) \tag{84}$$

where $Se(x)$ is the standard error of X, so that $\hat{\lambda} > max(X)$ or λ is the least upper bound

$$\hat{\lambda} = Sup(x_i) \tag{85}$$

where x is the value of a random variable X, that is $x \in X$. R code can also be used to estimate the parameters of LUB distribution using maxLik package in R and the code is shown in Appendix II.

4.3 *Hessian and Fisher's information matrices of LUB distribution*

From log likelihood in (81), we have the Hessian matrix, H of LUB distribution given as

$$H = \begin{pmatrix} -\frac{an}{\lambda} & -\frac{n}{\lambda} \\ -\frac{n}{\lambda} & \frac{an}{\lambda^2} \end{pmatrix}$$

The determinant of the Hessian matrix is $|H| = -\frac{n^2}{\lambda^2}\left(\frac{a^2+\lambda}{\lambda}\right)$ and the Fisher's information matrix is given by

$$-E(H) = \begin{pmatrix} \frac{an}{\lambda} & \frac{n}{\lambda} \\ \frac{n}{\lambda} & -\frac{an}{\lambda^2} \end{pmatrix}$$

The Hessian of LUB distribution has both positive and negative eigenvalues (see principal diagonal signs) then x is a saddle point for the function. Hence, the test is inconclusive. This implies that at a local minimum the Hessian is positive-semi-definite, and at a local maximum the Hessian is negative semi-definite. From the Morse theory, for two parameters, the determinant is used. In this case, the determinant is negative, then x is a local maximum. The Hessian matrices have many applications, they are used in large-scale optimization problems, image processing and computer vision.

4.4 *Writing the LUB distribution in the exponential class of family*

Let $y_1, ..., y_n$ denote n independent observations on a response. We treat y_i as a realization of a random variable Y_i. In the general linear model we assume that Y_i has a normal distribution with mean μ_i and variance σ^2.

$$Y_i \sim N(0, \sigma^2) \tag{86}$$

and we further assume that the expected value μ_i is a linear function of k predictors that take values $x_i^{'} = (x_{i1}, ..., x_{ip})$ for the i-th case, so that

$$\mu_i = x_i^{'}\beta \tag{87}$$

where β is a vector of unknown parameters. We will generalize this in two steps, dealing with the stochastic and systematic components of the model.

Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with a pdf of the exponential form:

$$f(x;\theta) = exp[K(x)p(\theta) + S(x) + q(\theta)] \tag{88}$$

with $K(x)$ and $S(x)$ being functions only of x, $p(\theta)$ and $q(\theta)$ being functions only of the parameter θ, and the support being free of the parameter θ. The pdf of LUB distribution is given as

$$f(x) = \frac{a}{\lambda^a}x^{a-1}, 0 \leq x \leq \lambda, a > 0$$

Taking log of both sides

$$lnf(x) = ln\left(\frac{a}{\lambda^a}x^{a-1}\right) \tag{89}$$

$$lnf(x) = lna - aln\lambda + (a-1)lnx \tag{90}$$

Taking exponential of both sides

$$f(x) = exp\{lna - aln\lambda + (a-1)lnx\} \tag{91}$$

The exponential class family is given by

$$f(x;a) = exp\{(a-1)lnx + lna - aln\lambda\} \tag{92}$$

where $K(x) = lnx$, $S(x) = 0$, $p(\theta) = a - 1$ and $q(\theta) = lna - aln\lambda$. .

Exponential criterion

Let $X_1, X_2, ..., X_n$ be a random sample from a LUB distribution written in the exponential form as:

THEOREM 4.3

$$f(x;a) = exp\{alnx - lnx + lna - aln\lambda\}$$

with a support that does not depend on a*. Then, the statistic:*

$$\sum_{i=1}^{n} lnx_i$$

is sufficient for θ*.*

Proof. Since $X_1, X_2, ..., X_n$ is a random sample, the joint pdf of $X_1, X_2, ..., X_n$ is, by independence:

$$f(x_1, x_2, ..., x_n; a) = f(x_1;a) \times f(x_2;a) \times ... \times f(x_n;a)$$

Inserting what we know to be the pdf in exponential form, we have:

$$f(x_1, x_2, ..., x_n; a) = exp\left[p(\theta)\sum_{i=1}^{n} K(x_i) + \sum_{i=1}^{n} S(x_i) + nq(\theta)\right] \tag{93}$$

So, substitute $K(x_i) = lnx_i$, $S(x_i) = -lnx_i$, $p(\theta) = a$ and $q(\theta) = lna - aln\lambda$ into (101) to have.

$$f(x_1, x_2, ..., x_n; a) = exp\left[a\sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n} lnx_i + n(lna - aln\lambda)\right]$$

which can be factored as:

$$f(x_1, x_2, ..., x_n; a) = \left\{exp\left[a\sum_{i=1}^{n} lnx_i + nlna - anln\lambda\right]\right\} \times \left\{exp\left[-\sum_{i=1}^{n} lnx_i\right]\right\}$$

■

We have factored the joint pdf into two functions, one (ϕ) being only a function of the statistic $\sum_{i=1}^{n} lnx_i$ and the other (h) not depending on the parameter a:

$$f(x_1, x_2, ..., x_n; a) = \phi\left[h\left(\sum_{i=1}^{n} lnx_i\right); a\right] \times h\left(x_1, x2, ..., x_n\right)$$

Therefore, the Factorization Theorem shows that $\sum_{i=1}^{n} lnx_i$ is a sufficient statistic for a.

If a random variable X has a distribution belonging to the exponential family, its pdf given in (88) can be expressed as

$$f(x; \theta, \phi) = c\left(x, \phi\right) exp\left(\frac{x\theta - b(\theta)}{d(\phi)}\right) \tag{94}$$

where θ and ϕ are parameters and $b(\theta)$, $d(\phi)$ and $c(x, \phi)$ are known functions. Also, note that

$$d(\phi) = \frac{\phi}{\pi}$$

where π is a known prior weight, usually 1. So, equation (94) can be written as

$$f(x; \theta, \phi) = c\left(x, \phi\right) exp\left(\frac{x\theta - b(\theta)}{\phi}\right)$$

It is also true that that if X has a distribution in the exponential family then it has mean and variance

$$E(X) = \mu = b'(\theta) \tag{95}$$

$$Var(X) = \sigma^2 = \left(\frac{\phi}{\pi}\right) b''(\theta) \tag{96}$$

where $b'(\theta)$ and $b''(\theta)$ are the first and second derivatives of $b(\theta)$. The LUB distribution can then be written in exponential form as

$$f(x; a) = exp\left(\frac{(a-1)lnx + lna - aln\lambda}{1}\right) \tag{97}$$

4.5 *Fitting LUB distribution to data*

In probability distribution, a proposed model is useful if it can model a particular set of data. This can be achieved by testing the proposed distribution with the data using Kolmogorov–Smirnov test and by testing the performance of the model with already established models using the Akaike information Criterion (AIC).

4.5.1 Kolmogorov–Smirnov test

In statistics, the Kolmogorov–Smirnov test (K–S test or KS test) is a non-parametric test of the equality of univariate continuous probability distributions that can be used to compare a data sample with a reference probability distribution (one-sample K–S test), or to compare two samples (two-sample K–S test) (Stephens, 1974).

The Kolmogorov–Smirnov statistic quantifies a distance between the empirical distribution function of the sample and the cumulative distribution function of the reference distribution, or between the empirical distribution functions of two samples (Stephens, 1974). The null distribution of this statistic is calculated under the null hypothesis that the sample is drawn from the reference distribution (in the one-sample case) or that the samples are drawn from the same distribution (in the two-sample case). In each case, the distributions considered under the null hypothesis are continuous distributions but are otherwise unrestricted. Marsaglia et al.(2003).

The K-S statistic is sensitive to differences in both location and shape of the empirical cumulative distribution functions of the two samples (Stephens, 1974).

The empirical distribution function F_n for n ordered observations X_i is defined by Kolmogorov (1933) and Smirnov (1948) as

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{[0,x]}(X_i), \tag{98}$$

where $I_{[0,x]}(X_i) = 1$, if $X_i \leq x$ and equal to 0 otherwise. Note that $I_{[0,x]}(X_i)$ is the indicator function. The Kolmogorov–Smirnov statistic for a given cdf $F(x)$ is

$$D_n = sup_x|F_n(x) - F(x)|. \tag{99}$$

where sup_x is the supremum of the set of distances between $F_n(x)$ and $F(x)$. Note that for LUB distribution $F(x)$ is defined in (12).

4.5.2 Akaike information criterion of LUB distribution

The Akaike information criterion (AIC) is based on information theory, which was formulated by a statistician called Hirotugu Akaike and it is widely used for statistical inference (Akaike, 1974). It estimates the relative quality of statistical models for a given data set. Given a class of selected models for the given data, AIC estimates the quality of one model in relative to others. Hence, AIC is a criterion for model selection. If a statistical model is used to represent the process that generated such data, the representation will almost never be exact, so losing some information by using the model to represent the process. AIC estimates the relative information lost by such model. The less a model loses such information, the higher the quality of that model. By estimating the information lost, AIC deals with the trade-off between the goodness of fit of such a model and the simplicity of the model(Boisbunon et al., 2014).

Supposed we have a statistical model of some data, say X. Let p be the number of estimated parameters in the model. Let $\hat{L}$ be the maximum value of the likelihood function for the model. Then the AIC value of the model proposed by (Akaike, 1973) is given as

$$AIC = 2p - 2ln(\hat{L}), \tag{100}$$

where $ln(\hat{L})$ for LUB distribution is given in (89). Now, put (89) in (108) to have the AIC for LUB distribution given by

$$AIC = 2\left\{2 - \left[nln(\hat{a}) - \hat{a}nln(\hat{\lambda}) + (\hat{a}-1)\sum_{i=1}^{n} lnx_i\right]\right\}, \tag{101}$$

where $\hat{a}$ and $\hat{\lambda}$ are the estimate of a and λ respectively in LUB distribution and $(x_i, i = 1\, to\, n) \in X$.

Given a set of candidate models for the data, the preferred model is the one with the minimum AIC value. Thus, AIC rewards goodness of fit, but it also includes a penalty that is an increasing function of the number of estimated parameters. The penalty discourages overfitting, because increasing the number of parameters in the model and almost always improves the goodness of the fit (Aho et al.2014).

4.6 *Transformation of data*

There are some data that need to be transformed or what some researchers call normalization of data, before it can be fitted well to LUB distribution. If the K-S p-value is very small, say less that 0.05, then we reject the null hypothesis and conclude that LUB distribution is not a good fit for the data. If such data is transformed, LUB distribution can be a good fit. Thus, we use the following transformation

$$y_i = \frac{x_i - Min(x_i)}{Max(x_i) - Min(x_i)}, \forall\, i \tag{102}$$

It is very obvious from (102) that the maximum of y is 1 and the minimum is zero, that is, $Max(y_i) = 1$ and $Min(y_i) = 0$. LUB distribution cannot be used to model any measured variable X that has zero as one of its data point. This is because the logarithm of zero is undefined. The numerator in (102) is the major reason why $Min(y_i) = 0$. To avoid this, we proposed the transformation for LUB distribution as

$$y_i = \frac{(x_i + SE(x_i)) - Min(x_i)}{Max(x_i) - Min(x_i)}, \forall\, i \tag{103}$$

where $SE(x_i)$ is the standard error of the variable X from the given data. In this case, the minimum value of Y cannot be zero, it will be strictly greater than zero, that is, $y_i > 0$, $\forall\, i$. Also, the maximum can no longer be 1 but will be greater than 1. In this case, the estimate of λ in Y will be greater than 1, since $SE(x_i)$ is a positive real number.

Since we cannot model any data set with zero included as one of the data points, we strictly advised that such type of data be transformed by the formula in equation (103), which will definitely be reduced to (104).

$$y_i = \frac{(x_i + SE(x_i))}{Max(x_i)}, \forall\, i \tag{104}$$

Better still, if data that contain zero are used, we can go ahead with equation (102) by removing all such points with zero entries and reduce the length of the data.

5. Results and discussion

5.1 *Simulation study*

A simulation study was carried out to estimate the parameters of the proposed distribution (LUB). The maximum likelihood estimation (MLE) of the parameters are in closed form. The values of a and λ can easily be estimated by the MLE estimators given in equations (83) and (84) respectively. The simulation study was carried out on sample sizes 20, 50, 100, 250 and 500, each replicated 1000 times for parameter values a = 0.4, 0.9, 2 and 5, and constant $\lambda = 10$. The R statistical package was used for the simulation study setting seed to 1234, that is, set.seed(1234). The results are setup in Microsoft Word table and sniped with the snipping tool as picture and saved with JPG extention. The actual values for the sample sizes 20, 50,100, 250 and 500, estimates of a and λ, absolute bias (AB) and standard error (SE) of the estimates are presented in Table 3.

Consistency of the parameter estimates

The estimates of parameters a and λ are consistent as shown by the values of absolute bias (AB) and the standard error (SE) computed as shown in Table 3. The AB and SE converges to zero as the sample size, n increases. The AB and SE trends are depicted in Fig. 7 to Fig. 10. The Figures show that they converge to zero as the n increased from 20 to 500. The Fig. 9 and Fig. 10 show that the SE converges faster for parameters a and λ than that of AB.

Table 3: The parameter estimates, absolute bias and standard error of LUB distribution for different values of a

Actual Values			Estimate		AB		SE	
a	λ	n	$\hat{a}$	$\hat{\lambda}$	$\hat{a}$	$\hat{\lambda}$	$\hat{a}$	$\hat{\lambda}$
0.4	10	20	0.4060	9.9687	0.006	0.0313	0.0032	0.0096
		50	0.4018	9.9727	0.0018	0.0273	0.0019	0.0053
		100	0.4013	9.9768	0.0013	0.0232	0.0013	0.0034
		250	0.4012	9.9763	0.0012	0.0237	0.0008	0.0021
		500	0.4012	9.9762	0.0012	0.0238	0.0006	0.0015
0.9	10	20	0.9135	9.9861	0.0135	0.0139	0.0072	0.0043
		50	0.9039	9.9878	0.0039	0.0122	0.0044	0.0024
		100	0.9029	9.9897	0.0029	0.0103	0.0030	0.0015
		250	0.9028	9.9894	0.0028	0.0106	0.0018	0.0009
		500	0.9027	9.9894	0.0027	0.0106	0.0013	0.0007
2	10	20	2.0300	9.9937	0.0300	0.0063	0.0160	0.0019
		50	2.0088	9.9945	0.0088	0.0055	0.0097	0.0011
		100	2.0064	9.9953	0.0064	0.0047	0.0067	0.0007
		250	2.0062	9.9952	0.0062	0.0048	0.0041	0.0004
		500	2.0061	9.9952	0.0061	0.0048	0.0028	0.0003
5	10	20	5.0750	9.9975	0.0750	0.0025	0.0399	0.0008
		50	5.0219	9.9978	0.0219	0.0022	0.0243	0.0004
		100	5.0159	9.9981	0.0159	0.0019	0.0167	0.0003
		250	5.0155	9.9981	0.0155	0.0019	0.0103	0.0002
		500	5.0152	9.9981	0.0152	0.0019	0.0071	0.0001

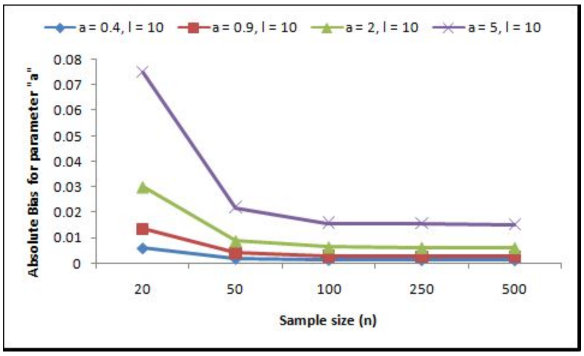


Figure 7: Absolute bias for parameter a

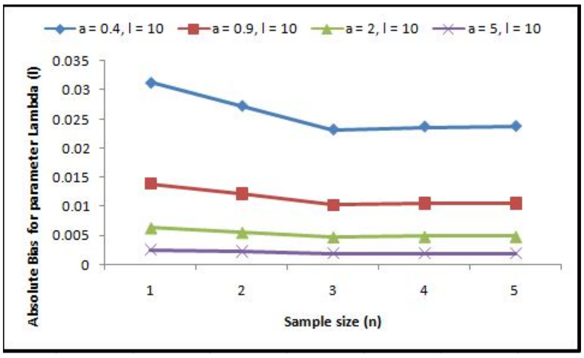


Figure 8: Absolute bias for parameter λ

5.2 *Application*

In this section, we shall present two empirical applications so that we can apply the tools developed in the previous sections to model the fitness of the data and compare the result to other existing distributions. The first application used the data set from Durbin and Koopman (2001), published in the work of Alzaatreh et al. (2016), represents the measurements of the annual flow of the Nile River at Ashwan from 1871–1970,

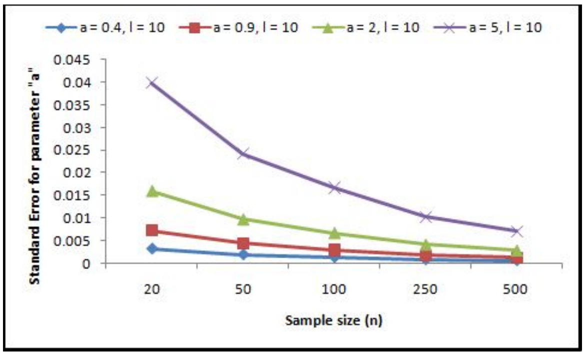


Figure 9: Standard error for parameter a

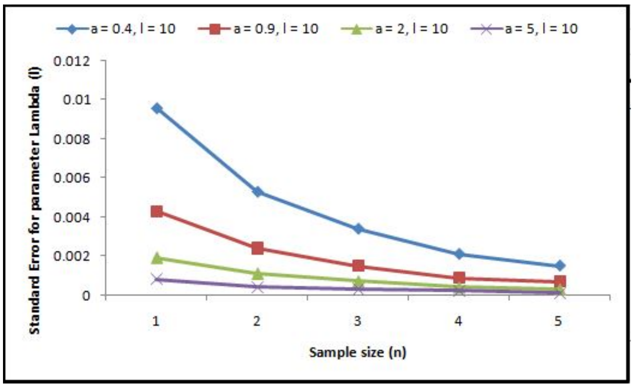


Figure 10: Standard error for parameter λ

shown in Table 4. The second data set is a real data from Mahmoudi and Sepahdar (2013), published in the work of Insuk et al. (2015), representing strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England, shown in Table 5. The R statistical package and Microsoft Excel were the major statistical packages used for the analysis. The data in Table 4 is approximately symmetric with

Table 4: The annual flow of the Nile River at Ashwan from 1871–1970

1120, 1160, 963, 1210, 1160, 1160, 813, 1230, 1370, 1140, 995, 935, 1110, 994, 1020, 960, 1180, 799,
958, 1140,1100, 1210, 1150, 1250, 1260, 1220, 1030, 1100, 774, 840, 874, 694, 940, 833, 701, 916, 692,
1020, 1050, 969,831, 726, 456, 824, 702, 1120, 1100, 832, 764, 821, 768, 845, 864, 862, 698, 845, 744,
796, 1040, 759, 781, 865,845, 944, 984, 897, 822, 1010, 771, 676, 649, 846, 812, 742, 801, 1040, 860,
874, 848, 890, 744, 749, 838, 1050,918, 986, 797, 923, 975, 815, 1020, 906, 901, 1170, 912, 746, 919,
718, 714, 740

skewness = 0.3175 and kurtosis = 2.6415. We transformed the data with the formula in (103), where X represents the original data and Y represents the transformed data. The transformed data Y was fitted to the LUB distribution and compared the results with Cauchy, gamma-Pareto proposed by Alzaatreh et al. (2012), beta-Cauchy distribution proposed by Alshawarbeh et al. (2013) and Gamma-Cauchy$\{$exponential$\}$ distribution proposed by Alzaatreh et al. (2016). The maximum likelihood estimates, the log-likelihood value, the AIC, the K-S test statistic, and the K-S statistic p-value for the fitted distributions are shown in Tables 6. The data in Table 5 is negatively skewed with skewness = -0.9220 and kurtosis = 1.1031. We fit the LUB distribution to the data in Table 5 and compared the fitness with some models in Insuk et al. (2015), which are Beta Exponentiated Exponential (BEE), Exponentiated Weibull (EW), Exponentiated Exponential (EE) and Weibull distributions by considering the p-value of K-S statistic. The maximum likelihood estimates of the parameters, the K-S statistics and the corresponding p-values for the fitted models are shown in Tables 7. The results in Tables 6 shows the LUB distribution, Gamma-CauchyExponential and beta-Cauchy provide an adequate fit to the survival time data while the LUB distribution provides the best fit using the p-value of the K-S. The result in Table 7 shows that the LUB distribution is the best using the K-S p-value. The LUB, Weibull and EW distributions fit the data but the LUB is the best considering that it has a p-value of 0.2921. Hence, the proposed LUB distribution competes favourably with existing distributions.

Table 5: The strengths of 1.5 cm glass fibers

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64
1.68	1.73	1.81	2.00	0.74	1.04	1.27	1.39	1.49
1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01	0.77
1.11	1.28	1.42	1.50	1.54	1.60	1.62	1.66	1.69
1.76	1.84	2.24	0.81	1.13	1.29	1.48	1.50	1.55
1.61	1.62	1.66	1.70	1.77	1.84	0.84	1.24	1.30
1.48	1.51	1.55	1.61	1.63	1.67	1.70	1.78	1.89

Table 6: Statistical evaluation of parameter estimates

Distribution	Cauchy	Gamma-Pareto	Beta-Cauchy	Gamma-Cauchy{Exponential}	LUB
Parameter Estimates	$\hat{c} = 139.3079$ $\hat{\theta} = 48.1262$	$\hat{\alpha} = 6.030$ $\hat{c} = 0.4497$ $\hat{\theta} = 10$	$\hat{\alpha} = 13.9274$ $\hat{\beta} = 4.5828$ $\hat{c} = 117.9055$ $\hat{\theta} = 27.0884$	$\hat{\alpha} = 16.1591$ $\hat{\beta} = 0.1027$ $\hat{\theta} = 110.1742$	$\hat{a} = 1.8567$ $\hat{\lambda} = 0.9997$
Log-likelihood	-674.4637	-696.7975	-653.4892	-654.3825	-17.8783
AIC	1352.9270	1397.5950	1314.9780	1314.7650	31.75661
K-S	0.1311	0.1705	0.0736	0.0637	0.0591
K-S p-value	0.0642	0.0060	0.6515	0.8120	0.8175

Table 7: Statistical evaluation of parameter estimates using K-S statistic

Distribution	BEE	EW	EE	Weibull	LUB
Parameter Estimates	$\hat{a} = 0.4021$ $\hat{b} = 31.4853$ $\hat{\alpha} = 24.8020$ $\hat{\theta} = 1.0976$	$\hat{\alpha} = 0.6712$ $\hat{\beta} = 7.2845$ $\hat{\theta} = 0.5820$	$\hat{\alpha} = 31.3485$ $\hat{\theta} = 2.6115$	$\hat{\beta} = 5.7807$ $\hat{\theta} = 0.6142$	$\hat{a} = 2.3506$ $\hat{\lambda} = 2.2400$
K-S	0.1999	0.1462	0.2291	0.1522	0.1246
K-S p-value	0.0130	0.1351	0.0027	0.1078	0.2921

We also present the comparison of the empirical cdf with each estimated cdf of the two data sets and this is depicted in Fig 11(a) and (b). The data 1 in Fig. 11(a) has a better fit than the data 2 in Fig. 11(b) because, the data 1 is a transformed data. Even the K-S p-value shows that LUB distribution fit the two data sets.

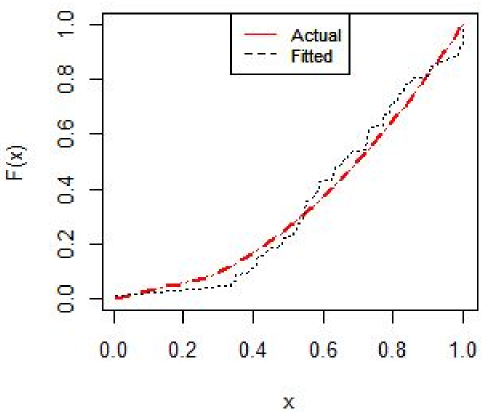


Figure 11(a): LUB fit data 1.

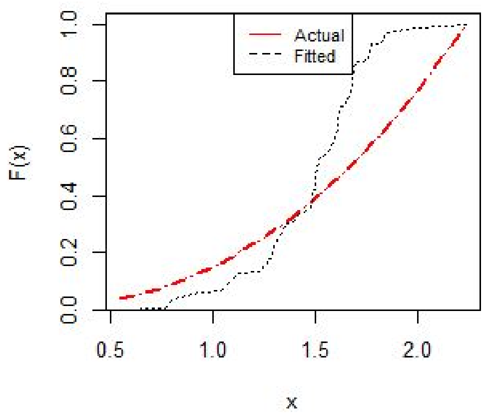


Figure 11(b): LUB fit to data 2

6. Conclusion

A Lambda Upper Bound (LUB) is proposed as a special case of Beta distribution. It is also referred to as two parameter power function distribution. It is derived by integrating beta distribution with parameters $\alpha = a$ and $\beta = 1$ on the closed interval $[0, \lambda]$ and equating the result to 1. We solved the resulting equation and derived the LUB distribution. The LUB distribution is similar to Beta, Uniform and Kumaraswamy distributions for some parameter values. It also has the LUB function, which is applicable to solve some kind of definite integral (or Riemann integral) problems. Several properties of the LUB distribution are studied including Shannon's entropy, order statistics and moments. For some parameter values, LUB distribution can be positively skewed if the values of parameter a is less than 1 ($a < 1$), or negatively skewed if $a > 1$ or uniform if $a = 1$. The simplicity and flexibility of the LUB distribution and the existence of the moments and closed form of its maximum likelihood estimates make this distribution an alternative to Beta, Uniform and Kumaraswamy distributions in situations where these distributions may not provide an adequate fit. The LUB distribution is tested on two data set. The result of the k-s p-value shows that it is better than some existing models in fitting some types of data. It will be very useful when data are transformed or when the data is bounded above. In real life, most data are non negative continuous and bounded above. LUB distribution can be used to model these types of data, at least for its simplicity and easy manipulation.

Acknowledgement

The authors wish to acknowledge the technical support of Professor Felix Famoye, Dr. Muminu Adamu and Professor Johnson Olaleru, and anonymous reviewers whose comments were useful in making the article strong.

References

Aho, K., Derryberry, D. and Peterson, T. (2014). Model selection for ecologists: the worldviews of AIC and BIC. Ecology, **95**: 631–636, doi:10.1890/13-1452.1.

Akaike, H. (1973). Information theory and an extension of the maximum likelihood principle. In: Petrov, B. N., and Csáki, F., 2nd International Symposium on Information Theory, Tsahkadsor, Armenia, USSR, Budapest: Akadémiai Kiadó, pp. 267–281.

Akaike, H. (1974). A new look at the statistical model identification. IEEE Transactions on Automatic Control, **19** (6): 716–723, doi:10.1109/TAC.1974.1100705, MR 0423716.

Akinsete, A., Famoye, F. and Lee, C. (2008). The beta-Pareto distribution. Statistics, **42**: 547–563.

Alshawarbeh, E., Famoye, F. and Lee, C. (2013). Beta-Cauchy distribution: some properties and applications. J. Stat. Theory Appl., **12**(4), 378–391.

Alzaatreh, A., Famoye, F. and Lee, C. (2012). Gamma-Pareto distribution and its applications. J. Mod. Appl. Stat. Methods, 11(1): 78–94.

Alzaatreh, A., Lee, C., Famoye, F. and Ghosh, I. (2016). The generalized Cauchy family of distributions with applications. Journal of Statistical Distributions and Applications, **3**(12), DOI 10.1186/s40488-016-0050-3.

Anake, T. A., Oguntunde, P. E. and Odetunmibi, O. A. (2015). On a fractional beta-exponential distribution. International Journal of Mathematics and Computation, 26(1).

Boisbunon, A., Canu, S., Fourdrinier, D., Strawderman, W. and Wells, M. T. (2014). Akaike's information criterion, Cp and estimators of loss for elliptically symmetric distributions. International Statistical Review, **82**: 422–439, doi:10.1111/insr.12052.

Bookstaber, R.M. and McDonald, J.B., (1987). A general distribution for describing security price returns. Journal of Business, **60**: 401–424.

Durbin, J. and Koopman, S.J. (2001). Time series analysis by state space methods. Oxford University Press, Oxford.

Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. Communications in Statistics- Theory and Methods, 31: 497-512.

Famoye, F., Akarawak, E. and Ekum, M. (2018). Weibull-normal distribution and its applications. Journal of Statistical Theory and Applications, 17(4): 719–727, doi: 10.2991/jsta.2018.17.4.12.

Insuk, T., Bodhisuwan, W. and Jaroengeratikun, U. (2015). A new mixed beta distribution and structural properties with applications. Songklanakarin J. Sci. Technol., 37(1): 97-108.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1995). Chapter 21: Beta Distributions. Continuous Univariate Distributions, Vol. 2 (2nd ed.). Wiley. ISBN 978-0-471-58494-0.

Kieschnick, R. and McCullough, B.D. (2003). Regression analysis of variates observed on (0, 1): Percentages, proportions and fractions. Statistical Modelling, **3**: 193–213.

Kolmogorov, A. (1933). Sulla determinazione empirica di una legge di distribuzione. G. Ist. Ital. Attuari., **4**: 83–91.

Kong, L., Lee, C. and Sepanski, J.H. (2007). On the properties of beta gamma distributions. Journal of Modern Applied Statistical Methods, **6**: 173–187.
Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. Journal of Hydrology, **46**(1–2): 79–88. doi:10.1016/0022-1694(80)90036-0.
Lee, C., Famoye, F. and Alzaatreh, A. (2013). WIREs Comp. Stat., **5**: 219–238.
Mahmoudi, E. and Sepahdar, A. (2013). Exponentiated Weibull-Poisson distribution: Model, properties and applications. Mathematics and Computers in Simulation, **92**: 76-97.
Marsaglia, G., Tsang, W.W. and Wang, J. (2003). Evaluating Kolmogorov's distribution. Journal of Statistical Software, **8**(18): 1–4.
McDonald, J.B. (1984). Some generalized functions for the size distribution of income, Econometrica, **52**: 647–663.
Nadarajah, S. and Kotz, S. (2004). The beta Gumbel distribution. Mathematics and Probability for Engineers, **10**: 323–332.
Nadarajah, S. and Kotz, S. (2005). The beta exponential distribution. Reliability Engineering and System Safety, **91**: 689–697.
Nassar, M. M. and Nada. N.K. (2011). The beta generalized Pareto distribution. Journal of Statistics–Advances in Theory and Applications, **6**: 1–17.
Nassar, M.M., and Nada, N.K (2012). A new generalization of the exponential geometric distribution. Journal of Statistics-Advances in Theory and Applications, **7**: 25-48.
Nassar, M.M. and Nada. N.K. (2013). A new generalization of the Pareto geometric distribution. Journal of Egyptian Mathematical Society, **21**(2): 148-155.
Nassarand, M. M. and Elmasry, A. (2012). A study of generalized Logistic distributions. Journal of Egyptian Mathematical Society, **20**(2): 126-133.
Park, S. Y. and Bera, A. K. (2009). Maximum entropy autoregressive conditional heteroskedasticity model. Journal of Econometrics, **150**(2): 219–230. doi:10.1016/j.jeconom.2008.12.014.
Pescim, R.R., Demetrio, C.G.B., Cordeiro, G.M., Ortega, E.M. M. and Urbano, M. R. (2010). The beta generalized half-normal distribution. Computational Statistics and Data Analysis, **54**: 945–957.
Smirnov, N. (1948). Table for estimating the goodness of fit of empirical distributions. Annals of Mathematical Statistics, **19**: 279–281. doi:10.1214/aoms/1177730256.
Souza, W.B. , Santos, A.H.S. and Cordeiro, G.M. (2010). The beta generalized exponential distribution. Journal of Statistical Computation and Simulation, **80**; 159–172.
Stephens, M. A. (1974). EDF statistics for goodness of fit and some comparisons. Journal of the American Statistical Association, **69**(347): 730–737. doi:10.2307/2286009.
Westfall, P.H. (2014). Kurtosis as Peakedness, 1905 - 2014. R.I.P. Am Stat., **68**(3): 191–195. doi:10.1080/00031305.2014.917055. PMC 4321753. PMID 25678714.

Appendix

Appendix I. Code for LUB Functions

```
# LUB Distribution
rm(list=ls())
set.seed(1234)
N = 1000
l = 10 # l represents parameter lambda
a1 = 0.8
a2 = 1.0
a3 = 2.0
a4 = 5.0
p = sort(runif(N,0,1))
x = l * p^(1/a2) #Quantile function
x1 = l * p^(1/a1)
x2 = l * p^(1/a2)
x3 = l * p^(1/a3)
x4 = l * p^(1/a4)

#PDF plot
fx1 = (a1/l^(a1)) * x^(a1 - 1)
fx2 = (a2/l^(a2)) * x^(a2 - 1)
fx3 = (a3/l^(a3)) * x^(a3 - 1)
fx4 = (a4/l^(a4)) * x^(a4 - 1)
plot(x, fx1, lty = 2, lwd = 2, col = "red", xlab = "x", ylab = "f(x)", type = "l", pch = 19)
lines(x, fx1, pch = 18, lty = 3, col = "red")
lines(x, fx2, pch = 18, lty = 3, col = "black")
lines(x, fx3, pch = 18, lty = 3, col = "blue")
lines(x, fx4, pch = 18, lty = 3, col = "green")
```

$legend("top", legend = c("a = 0.8, lambda = 10.0", "a = 1.0, lambda = 10.0", "a = 2.0, lambda = 10.0", "a = 5.0, lambda = 10.0"), col = c("red", "black", "blue", "green"), lty = 1:2, cex = 0.8)$

\# **CDF Plots**
$Fx1 = (x/l)\hat{}a1$
$Fx2 = (x/l)\hat{}a2$
$Fx3 = (x/l)\hat{}a3$
$Fx4 = (x/l)\hat{}a4$
$plot(x, Fx1, lty = 2, lwd = 2, col = "red", xlab = "x", ylab = "F(x)", type = "l", pch = 19)$
$lines(x, Fx1, pch = 18, lty = 3, col = "red")$
$lines(x, Fx2, pch = 18, lty = 3, col = "black")$
$lines(x, Fx3, pch = 18, lty = 3, col = "blue")$
$lines(x, Fx4, pch = 18, lty = 3, col = "green")$
$legend("top", legend = c("a = 0.8, lambda = 10.0", "a = 1.0, lambda = 10.0", "a = 2.0, lambda = 10.0", "a = 5.0, lambda = 10.0"), col = c("red", "black", "blue", "green"), lty = 1:2, cex = 0.8)$

\#**Survival function plots**
$Sx1 = 1 - Fx1$
$Sx2 = 1 - Fx2$
$Sx3 = 1 - Fx3$
$Sx4 = 1 - Fx4$
$plot(x, Sx1, lty = 2, lwd = 2, col = "red", xlab = "x", ylab = "S(x)", type = "l", pch = 19)$
$lines(x, Sx1, pch = 18, lty = 3, col = "red")$
$lines(x, Sx2, pch = 18, lty = 3, col = "black")$
$lines(x, Sx3, pch = 18, lty = 3, col = "blue")$
$lines(x, Sx4, pch = 18, lty = 3, col = "green")$
$legend("bottomleft", legend = c("a = 0.8, lambda = 10.0", "a = 1.0, lambda = 10.0", "a = 2.0, lambda = 10.0", "a = 5.0, lambda = 10.0"), col = c("red", "black", "blue", "green"), lty = 1:2, cex = 0.8)$

\#**Hazard function plots**
$hx1 = fx1/Sx1$
$hx2 = fx2/Sx2$
$hx3 = fx3/Sx3$
$hx4 = fx4/Sx4$
$plot(x, hx1, lty = 2, lwd = 2, col = "red", xlab = "x", ylab = "h(x)", type = "l", pch = 19)$
$lines(x, hx1, pch = 18, lty = 3, col = "red")$
$lines(x, hx2, pch = 18, lty = 3, col = "black")$
$lines(x, hx3, pch = 18, lty = 3, col = "blue")$
$lines(x, hx4, pch = 18, lty = 3, col = "green")$
$legend("topleft", legend = c("a = 0.8, lambda = 10.0", "a = 1.0, lambda = 10.0", "a = 2.0, lambda = 10.0", "a = 5.0, lambda = 10.0"), col = c("red", "black", "blue", "green"), lty = 1:2, cex = 0.8)$

\#**Cumulative hazard function plots**
$Hx1 = -log(Sx1)$
$Hx2 = -log(Sx2)$
$Hx3 = -log(Sx3)$
$Hx4 = -log(Sx4)$
$plot(x, Hx1, lty = 2, lwd = 2, col = "red", xlab = "x", ylab = "H(x)", type = "l", pch = 19)$
$lines(x, Hx1, pch = 18, lty = 3, col = "red")$
$lines(x, Hx2, pch = 18, lty = 3, col = "black")$
$lines(x, Hx3, pch = 18, lty = 3, col = "blue")$
$lines(x, Hx4, pch = 18, lty = 3, col = "green")$
$legend("topleft", legend = c("a = 0.8, lambda = 10.0", "a = 1.0, lambda = 10.0", "a = 2.0, lambda = 10.0", "a = 5.0, lambda = 10.0"), col = c("red", "black", "blue", "green"), lty = 1:2, cex = 0.8)$

\#**Histograms**
$par(mfrow = c(2, 2))$
$hist(x1, main = "a = 0.8, lambda = 10", xlab = "x")$
$hist(x2, main = "a = 1.0, lambda = 10", xlab = "x")$
$hist(x3, main = "a = 2.0, lambda = 10", xlab = "x")$
$hist(x4, main = "a = 5.0, lambda = 10", xlab = "x")$

Appendix II. Log-Likelihood function with maxLik package

```
library(maxLik)
lubd = function(param, y){
a = param[1]
l = param[2]
L = n * log(a) − n * a * log(l) + (a − 1) * sum(log(y))
return(L)
}
Est = maxLik(lubd, y = y, start = c(1, 1))
```