BENIN JOURNAL OF STATISTICS
ISSN 2682-5767
Vol. 3, pp. 116– 127 (2020)

Modeling Two Stock Prices using Diffusion Process

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Abstract. In real life applications, the parameters of a stochastic differential equation (SDE) are unknown and need to be estimated. In most cases what is available is only sampled data of the process at discrete times. It is a common practice to use the discretization of the original continuous time process for the modeling. SDEs have solutions in continuous-time called diffusion process. The methodology as applied to stock price involves a discrete time stochastic model for the dynamical system under study. For a small time interval, Δt, the possible changes with their corresponding transition probabilities are determined. The expected change and the covariance matrix for the change are determined for the discrete time stochastic process. The system of SDEs is obtained by letting the expected change divided by Δt,- be the drift coefficient and the square root of the covariance matrix divided by Δt,- be the diffusion coefficient. The SDE model is inferred by similarities in the forward Kolmogorov equations between the discrete and the continuous stochastic processes. The resulting SDE model with estimated drift and volatility parameters is solved using the multi-dimensional Euler-Maruyama scheme for SDEs as implemented using the R packages 'Sim.DiffProc' and 'yuima'.

Keywords: Markov process, stochastic differential equation, transition probabilities, diffusion process, dynamical system, drift, volatility.

Published by: Department of Statistics, University of Benin, Nigeria

1. Introduction

A dynamical system is a system that describes the time dependence of a point in a geometrical space. Examples of such a system include traffic at a particular street light, the price of equities and stocks, the number of vehicles arriving at a filling station for fuel over a period, etc. Dynamical systems can be represented by the rules of evolution that describe the future state of the system from the current state. A dynamical system can be deterministic if only one future state is determined by the current state of the system or stochastic when future state is determined by random events or states. Most physical processes in the real world involve a random element in their structure. Thus a stochastic process can be described as a statistical phenomenon that evolves in time according to probabilistic laws (Chatfield, 2004).

Dynamical systems are usually described using differential equations which give the time derivatives for the system. Ordinary differential equations are used for system in finite dimensional space, but there are extensions to infinite-dimensional space, in which case the differential equations are partial differential equations. Dynamical system perspective to partial differential equations has begun to gain popularity. Unlike deterministic models that are based on ordinary differential equations, with a unique solution for each appropriate initial condition, stochastic differential equations (SDEs) have solutions that are in continuous-time. A major motivation for SDEs in modeling real world dynamical systems is that there is a nonlinear dependence of the level of the series on previous data points (Allen, 2007).

Casdagli (1991), in his work on forecasting algorithm, attempts to "bridge the gap" between the stochastic and deterministic approaches were made. The forecasting algorithm was applied to a variety of experimental and naturally occurring time series data, with a view to investigating whether a dataset exhibits low-dimensional chaotic behaviour, as opposed to high-dimensional or stochastic behaviour.

Tiberio (2004) noted that the change in a system could be done from a stochastic or deterministic perspective. Models based on deterministic differential equation are more likely to be implemented by sociologists

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and psychologists, whereas models based on stochastic differential equation are more regularly implemented by econometricians (Tiberio, 2004).

In mathematical finance traditional models are based on stochastic processes which are variables that evolve over time in a way that is at least in part random. Prices of financial assets are believed to have a random behavior fully or partly with some known mathematical and distributional properties. Stochastic behavior is usually driven by Brownian motion (or Wiener process). Brownian motion is a Markovian stochastic process with independent, stationary and normally distributed increments (McNicholas et al., 2012).

Wang (2015) examined dynamical models of stock prices based on technical trading rules, referred to as the moving average rules. The work showed that the price dynamical model has an infinite number of equilibriums, but all these equilibriums were unstable and that although the price was chaotic, the volatility converged to some constant very quickly at the rate of the Lyapunov exponent.

The chaotic nature of stock prices as revealed by Wang (2015) makes it crucial in examining the models used to fit stocks prices. Damptey (2017) looked at the appropriateness of the Geometric Brownian motion model for forecasting stock prices by comparing simulated data sets with market prices of some companies. The findings showed that the Geometric Brownian Motion (GBM) model is a more appropriate model for forecasting stock prices on the Ghana stock exchange.

Reddy and Clinton (2016) examined the evidence of simulating stock price using Geometric Brownian Motion from Australian companies. The results showed that over all time horizons the chances of a stock price simulated using GBM moving in the same direction as real stock prices was a little greater than 50 percent.

Reddy and Clinton (2016) noted that the correlation coefficient between simulated stock prices and actual stock prices, mean absolute percentage error and directional movements of simulated and actual stock prices may be used to test the validity of the model.

Maruddani and Trimono (2018) expanded the GBM model to carter for portfolio of stocks by using multidimensional GBM and they reported that the model gave a less mean absolute percentage error than that of GBM for single stock. They simulated stock price from the Indonesian Exchange (IDX) Top Ten Blue 2016 for two companies and reported the mean absolute percentage error value was less than 10% in favour of the multidimensional GBM. This means that the method is highly accurate for the model.

With the findings of Wang (2015) about the chaotic behavior of stock prices and the Markov property of SDE which lends itself to the concept of short term predictability of chaotic systems, this becomes a motivation to examine diffusion process in modeling a dynamical system (stock prices) using discrete stochastic differential equations for the modeling process. Hence, the main aim of the study is to model the dynamics of two stock prices using a stochastic differential equation of the Ito type, given the discrete-time realisations from the process. Even though the study considered a continuous-time process (stock price), the stack reality is that realizations from the continuous-time processes are taken in discrete times. So the approach is to describe the continuous-time process using the discrete-time observations

2. Materials and method

2.1 *Chapman Kolmogorov formula*

For discrete time stochastic processes with $\tau = (t_0, t_1, t_2, \dots)$ defining a set of discrete times and $X(t_0), X(t_1), X(t_2), \dots$ defining the sequence of random variables each on the sample space Ω. This system may describe, for instance the evolution of stock price over the discrete times $t_0, t_1, t_2, \dots$. This study relies on the Markov property that the present value of a random variable $X(t_n)$ *only* determines the future value $X(t_{n+1})$ and such systems (X_n), are said to be Markov processes. Markov processes are usually useful in developing the SDE models for biological and financial systems, the latter of which this study focuses on.

Given the time interval $\Delta t = 1/N$, for $t_i = i\Delta t$, $i = 0, 1, 2, \dots, N$ and $X_i = X(t_i)$ with $X_0 = 0$, there is a discrete random variable δ, that takes on values $-\beta, 0, \beta$, representing nominal values defining the movement of the variable $X(t_n)$ at any time t_{n+1}. For the stock price, the nominal values $-\beta, 0, \beta$, represents a loss, no change or a gain in stock price at t_{n+1}.

One property of the Markov process is the relation known as the Chapman-Kolmogorov formula

$$p_{i,j}^{(l+n)} = \sum_{m=0}^{\infty} p_{i,m}^{(l)} p_{m,j}^{(n)} \; \forall \; l, n \geq 0$$

where $p_{i,j} = P(X_{n+1} = j \mid X_n = i)\, i \geq 0,\ j \geq 0$ defines the transition probabilities for discrete times $t_n = n\Delta t$ so that $t_{n+k} = t_n + t_k$ (Allen, 2007).

This property for a discrete stochastic process is contained in the forward Kolmogorov equation which is of interest in developing models with stochastic differential equation herein as described by Allen (2007). Let $t_i = i\Delta t$ for $i = 0, 1, \ldots, N$ and let $x_j = j\delta$, for $j = ..., -2, -1, 0, 1, 2, \ldots$. Let X_0 be given. Define the transition probabilities of the discrete random variable δ representing the change in the discrete-time stochastic process X_n per time by the following:

$$p_{i,k}(t) = \begin{array}{ll} r(t, x_i)\frac{\Delta t}{\delta^2}, & for\ k = i + 1 \\ 1 - r(t, x_i)\frac{\Delta t}{\delta^2} - s(t, x_i)\frac{\Delta t}{\delta^2}, & for\ k = i \\ s(t, x_i)\frac{\Delta t}{\delta^2}, & for\ k = i - 1 \end{array}$$

where r and s are smooth nonnegative functions. If ΔX is the change in the stochastic process at time t fixing $X(t) = x_i$, then the mean change $E(\Delta X) = (r(t, X) - s(t, X))/\Delta t/\delta$ and variance in change $Var(\Delta X) = (r(t, X) - s(t, X))\Delta t$. It is assumed that $\Delta t/\delta^2$ is so small that $1 - r(t, x_i)\Delta t/\delta^2 - s(t, x_i)\Delta t/\delta^2$ is positive. Let $p_k(t) = P(X(t) = x_k)$ be the probability distribution at time t. Then, $p_k(t + \Delta t)$ satisfies

$$\begin{aligned} p_k(t + \Delta t) = p_k(t) + [p_{k+1}(t)s(t, x_{k+1}) - p_k(t)(r(t, x_k) + s(t, x_k)) + \\ p_{k-1}(t)r(t, x_{k-1})]\Delta t/\delta^2 \end{aligned} \tag{1}$$

As $\Delta t \to 0$, the discrete stochastic process approaches a continuous-time process. As $\Delta t \to 0$, $p_k(t)$ satisfies the initial-value problem:

$$\frac{dp_k(t)}{dt} = -\left(\frac{p_{k+1}(t)a(t,x_{k+1}) - p_{k-1}(t)a(t,x_{k-1})}{2\delta}\right) + \left(\frac{p_{k+1}(t)b(t,x_k+1) - 2p_k(t)b(t,x_k) + p_{k-1}(t)b(t,x_k-1)}{2\delta^2}\right) \tag{2}$$

for $k = ..., -2, -1, 0, 1, 2, \ldots$ and $(p_k(0))_{k=-m}^{m}$ are known. Equation (2) is the *forward Kolmogorov equation* for the continuous-time stochastic process which approximates the partial differential equation

$$\frac{\partial p(t, x)}{\partial t} = -\frac{\partial(a(t, x)p(t, x))}{\partial x} + \frac{1}{2}\frac{\partial^2(b(t, x)p(t, x))}{\partial x^2} \tag{3}$$

and corresponds to a *diffusion process* having the stochastic differential equation

$$dX(t) = a(t, X)dt + \sqrt{b(t, X)}dW(t) \tag{4}$$

An in-depth presentation of this process can be found in Allen (2007).

There exists a relationship between the discrete stochastic process defined by (3) and the continuous process defined by (4). For small Δt and δ, the probability distribution of the solution to (4) will be approximately the same as the probability distribution of solutions to the discrete stochastic process (Allen, 2007). A realistic discrete stochastic process model for the dynamical system under investigation in this study can thus be constructed by developing a discrete stochastic differential equation model. As time is made continuous (for small Δt and δ), then the solution of the stochastic differential equation approximates the dynamical system under study.

2.2 *Diffusion process model for stock price*

It is important to note that certain assumptions are made here in the derivation of the model for stock prices and these include:

- The process for stock price is stochastic rather than deterministic and a finite Δt produces a discrete stochastic model.
- The form assumed for the probabilities of the possible price changes over a small time step is that the probability of a change in one stock price is proportional to the stock price.

- For a simultaneous change in both stock prices, it is assumed that the probability of the change is proportional to the product of the two stock prices.
- The time interval Δt is sufficiently short such that the probability of more than one change in the stock prices is small and the probability that change in both stocks is zero is positive.
- Large jumps in stock prices caused by sudden major changes in the financial environment are not considered in this model development.

Two stocks are considered along with a fixed-interest money market account. The results can be readily generalized to a system of n stocks. Stock prices are assumed to change in a small time interval Δt.

Figure 1: Schematic illustration of the two stock Process

Let $\Delta S = [\Delta S_1, \Delta S_2]^T$ be the change in the two stock prices over a short time step Δt. It is necessary to find the mean and the covariance matrix for the change ΔS. Neglecting multiple changes in time Δt which have probabilities of order Δt^2, there are nine possibilities for ΔS in time Δt. A stock price may, for example, change by losing one unit (-1), remain stable (0) or gain one unit. For the two stocks considered here, $\Delta S = [1, 1]^T$ represents a one unit gain in both stocks, $\Delta S = [-1, -1]^T$ represents a one unit loss in both stocks, $\Delta S = [0, 0]^T$ represents no change in both stocks and $\Delta S = [-1, 0]^T$ represents a one unit loss in stock 1 and no change stock 2. These outcomes are given in the first column of Table 1. The parameters

Table 1: Possible changes in the stock prices with the corresponding probabilities

Change $[\Delta S_1, \Delta S_2]^T$	Probability
$[\Delta S_1, \Delta S_2]^T_1 = [1, 0]^T$	$p_1 = b_1 S_1 \Delta t$
$[\Delta S_1, \Delta S_2]^T_2 = [-1, 0]^T$	$p_2 = d_1 S_1 \Delta t$
$[\Delta S_1, \Delta S_2]^T_3 = [0, 1]^T$	$p_3 = b_2 S_2 \Delta t$
$[\Delta S_1, \Delta S_2]^T_4 = [0, -1]^T$	$p_4 = d_2 S_2 \Delta t$
$[\Delta S_1, \Delta S_2]^T_5 = [1, 1]^T$	$p_5 = m_{22} S_1 S_2 \Delta t$
$[\Delta S_1, \Delta S_2]^T_6 = [1, -1]^T$	$p_6 = m_{21} S_1 S_2 \Delta t$
$[\Delta S_1, \Delta S_2]^T_7 = [-1, 1]^T$	$p_7 = m_{12} S_1 S_2 \Delta t$
$[\Delta S_1, \Delta S_2]^T_8 = [-1, -1]^T$	$p_8 = m_{11} S_1 S_2 \Delta t$
$[\Delta S_1, \Delta S_2]^T_9 = [0, 0]^T$	$p_9 = 1 - \sum_{i=1}^{8} p_i$

(Allen, 2007).

$b_1, d_1, b_2, d_2, m_1 1, m_1 2, m_2 1 and m_2 2$ in Table 1 define the rates at which stocks experience individual gains or losses or experience simultaneous gains and/or losses. For example, $b_i S_i \Delta t$ is the probability that stock i for $i = 1$ or 2 has a gain of one unit in time interval Δt, $m_2 2 S_1 S_2 \Delta t$ is the probability that both stocks experience a gain in time interval Δt, and $m_1 2 S_1 S_2 \Delta t$ is the probability that stock 1 has a loss and stock 2 has a gain in time interval Δt. These probabilities given in second column of Table 1 are defined as the ratio of the frequency of change ΔS and the total number of observations N, for instance,

$$p_1 = b_1 = \frac{Number of times [1,0]^T was observed}{Total number of observations}$$

and

$$p_5 = m_{22} = \frac{Number of times [1,1]^T was observed}{Total number of observations}$$

Using the above expressions for p_i and ΔS_i, the expectation vector and the covariance matrix for the change ΔS can be derived as follows. Neglecting the terms of order $(\Delta t)^2$, it follows that

$$E(\Delta S) = \sum_{j=1}^{9} p_j [\Delta S_1, \Delta S_2]^T = b_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} S_1 + d_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} S_1 + b_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} S_2$$

$$+ d_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} S_2 + m_{22} \begin{pmatrix} 1 \\ 1 \end{pmatrix} S_1 S_2 + m_{21} \begin{pmatrix} 1 \\ -1 \end{pmatrix} S_1 S_2 + m_{12} \begin{pmatrix} -1 \\ 1 \end{pmatrix} S_1 S_2$$

$$+ m_{11} \begin{pmatrix} -1 \\ -1 \end{pmatrix} S_1 S_2 + m_{00} \begin{pmatrix} 0 \\ 0 \end{pmatrix} S_1 S_2$$

$$= \begin{pmatrix} b_1 \\ 0 \end{pmatrix} S_1 + \begin{pmatrix} -d_1 \\ 0 \end{pmatrix} S_1 + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} S_2 + \begin{pmatrix} 0 \\ -d_2 \end{pmatrix} S_2 + \begin{pmatrix} m_{22} \\ m_{22} \end{pmatrix} S_1 S_2$$

$$+ \begin{pmatrix} m_{21} \\ -m_{21} \end{pmatrix} S_1 S_2 + \begin{pmatrix} -m_{12} \\ m_{12} \end{pmatrix} S_1 S_2 + \begin{pmatrix} -m_{11} \\ -m_{11} \end{pmatrix} S_1 S_2 + \begin{pmatrix} 0 \\ 0 \end{pmatrix} S_1 S_2$$

$$= \begin{pmatrix} (b_1 - d_1) S_1 + (m_{22} + m_{21} - m_{12} - m_{11}) S_1 S_2 \\ (b_2 - d_2) S_2 + (m_{22} - m_{21} + m_{12} + m_{11}) S_1 S_2 \end{pmatrix}$$

$$E(\Delta S) = \mu(t, S_1, S_2) \Delta t$$

and

$$E(\Delta S (\Delta S)^T)) = \sum_{j=1}^{9} p_j \Delta S_j (\Delta S_j)^T = \begin{bmatrix} c_1 + c_2 + c_3 & c_2 - c_3 \\ c_2 - c_3 & c_2 + c_3 + c_4 \end{bmatrix} \Delta t$$

$$V(t, S_1, S_2) \Delta t$$

where $c_1 = (b_1 + d_1) S_1$, $c_2 = (m_{22} + m_{11}) S_1 S_2, c_3 = (m_{21} + m_{12}) S_1 S_2$, and $c_4 = (b_2 + d_2) S_2$. As the product $E(\Delta S)(E(\Delta S))^T$is of order $(\Delta t)^2$, the covariance matrix V is set to equal to $E(\Delta S (\Delta S)^T)/\Delta t$ (Allen, 2007).

Now V is positive definite and hence has a positive definite square root. Let $B = (V)^{1/2}$, as $\Delta t \to 0$, the probability distribution of the stock prices approximates the probability distribution of solutions to the Ito stochastic differential equation system (Allen, 2007):

$$dS(t) = \mu(t, S_1, S_2)dt + B(t, S_1, S_1)dW(t) \tag{5}$$

with $S(0) = S_0$ and where $W(t)$ is the two-dimensional Wiener process $W(t) = [W_1(t), W_2(t)]^T$. Equation (5) is a system of stochastic differential equations that describe the dynamics of the stock prices with

$$\mu(t, S_1, S_2) = \begin{bmatrix} (b_1 - d_1)S_1 + (m_{22} + m_{21} - m_{12} - m_{11})S_1S_2 \\ (b_2 - d_2)S_2 + (m_{22} - m_{21} + m_{12} + m_{11})S_1S_2 \end{bmatrix} \tag{6}$$

with $c_1 = (b_1 + d_1)S_1,\ c_2 = (m_{22} + m_{11})S_1S_2,\ c_3 = (m_{21} + m_{12})S_1S_2,\ c_4 = (b_2 + d_2)S_2$ and

$$B(t, S_1, S_2) = \tfrac{1}{d} \begin{bmatrix} c_1 + c_2 + c_3 + w & c_2 - c_3 \\ c_2 - c_3 & c_2 + c_3 + c_4 + w \end{bmatrix} \tag{7}$$

where $w = \sqrt{(c_1 + c_2 + c_3)(c_2 + c_3 + c_4)(c_2 - c_3)^2}$ and $d = \sqrt{c_1 + 2c_2 + 2c_3 + c_4 + 2w}$.

As Allen (2007) noted, the standard geometric Brownian motion model has its drift and diffusion coefficients proportional to the stock price. The drift and diffusion coefficients for a single stock of the model here, on the other hand, is a linear function of the stock price and similar to an affine model as described in Chernov et al., (2003).

3. Results and discussion

3.1 *Data description and visualisation*

In other to characterize the drift and volatility which will form the coefficients of the stochastic differential equations, the daily stock prices of two selected stocks Coca-cola and Pepsi for the period of 2009-07-06 to 2019-07-05 were collected from Yahoo Finance Online, for the modeling process.

Figure 2 shows the plot of the stock price dataset for Coca-cola and Pepsi. Figure 2-A shows the plot of the stocks of Pepsi and Figure 2-B shows the plot of the stocks of Coca-cola for the period the data was observed. From these plots it can be observed that the stock prices of Pepsi within the period is above the stock price of Coca-cola and both stocks have appreciated significantly within the period. However, further analysis of the stocks and modeling will give details not obvious from these two plots (Figure 2-A and Figure 2-B). Figure 2-C shows the plot of both stocks and the difference observed can be attributed to the fact that, while absolute price is important (pricey stocks are difficult to purchase, which affects not only their volatility but the ability to trade that stock), when trading, more concern is on the relative change of an asset rather than its absolute price. Pepsi's stocks from the dataset are more expensive than Coca-cola's stocks in terms of their stock prices on stock exchange, but not on consumer prices in the market and this difference makes Coca-cola's stocks appear much less volatile than they truly are (that is, their price appears to not deviate much as can be seen in Figure 2-C).

One solution would be to use two different scales when plotting the data; one scale will be used by Coca-cola stocks and the other by Pepsi stocks. This results in the plot in Figure 2-D, but sometimes this solution can be difficult to implement well and it can lead to misinterpretation, and may not be read easily. A transformation the data into a more useful format may give better information than already obtained. One transformation would be to consider the stocks' return since the beginning of the period of interest, that is, to plot the returns defined as:

$$return_{t,0} = \frac{price_t}{price_0}$$

As the interest in this study is in the change of the stocks per day, a plot of this change will also be made by plotting the log-difference of the stocks, which can be interpreted as the percentage change in a stock but

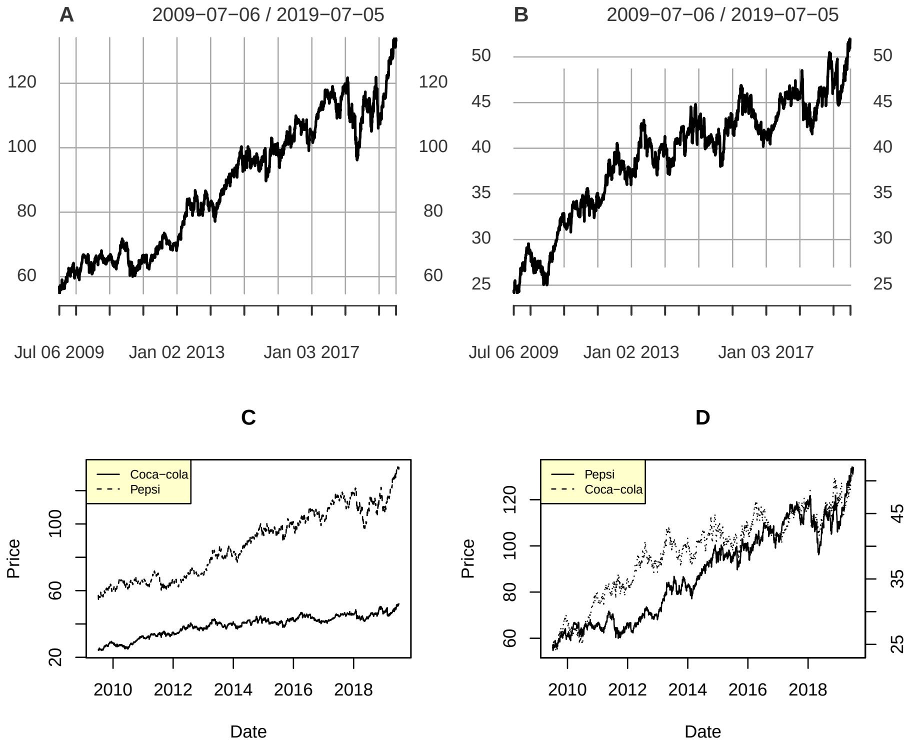

Figure 2: Various Plots of Stock Prices of Coca-cola and Pepsi

does not depend on the denominator of a fraction, when comparing day t to day $t+1$, with the formula:

$$change_t = \lg(price_{t+1}) - \lg(price_t)$$

Figure 3 shows the plots of the returns and log-differences of both stock prices. From Figure 3-A, the profitability of each stock since the beginning of the period can be seen. The stocks of Coca-cola was more profitable than the stock of Pepsi between the period 2011 and 2014, but the stock of Coca-cola dropped below the stocks of Pepsi between 2017 and the end of the period 2019 as the stocks of Pepsi was more profitable. This trend can be further seen in the log-difference plot as from the beginning of the period the log-difference of the daily change in Pepsi stock were higher than those of Coca-cola stock and vice versa. Furthermore, it can be seen that these stocks are correlated; they generally move in the same direction, a fact that was difficult to see in Figure 2-C.

3.2 *Stochastic differential modeling of Coca-Cola and Pepsi stocks data*

The data from 3/01/2017 to 5/07/2019 for stocks of Coca-cola (S_1) and Pepsi (S_2) for 630 days period were used for the modeling. The corresponding transition probabilities are given in Table 2, which also shows the daily simultaneous change in S_1 and S_2. The methodology described in Section 2.2 is applied to the set of data for both stock prices to model the stochastic process. The second column in Table 2 shows the frequency of each possible simultaneous change in both stocks.

The procedure involved for obtaining the number of occurrence was that the prices of both stocks for the future day $t+1$, were compared with the prices of the current day t, and if a stock gained in price, the nominal values 1, was taken if a stock lost in price, -1 was noted and if the price was unchanged, 0 was recorded. This process is repeated for both stocks each day from day number 1 to number 630. At the end, the number of occurrence for each of the nine possible changes for both stocks is recorded as given in the second column of Table 2, out of the total of 630 days. The probability column of Table 2 is as defined in Section 2.2 and is the relative frequency of the number of occurrence and the total number of periods the observations were taken (630). From Table 2, the parameters of drift coefficient and volatility are computed

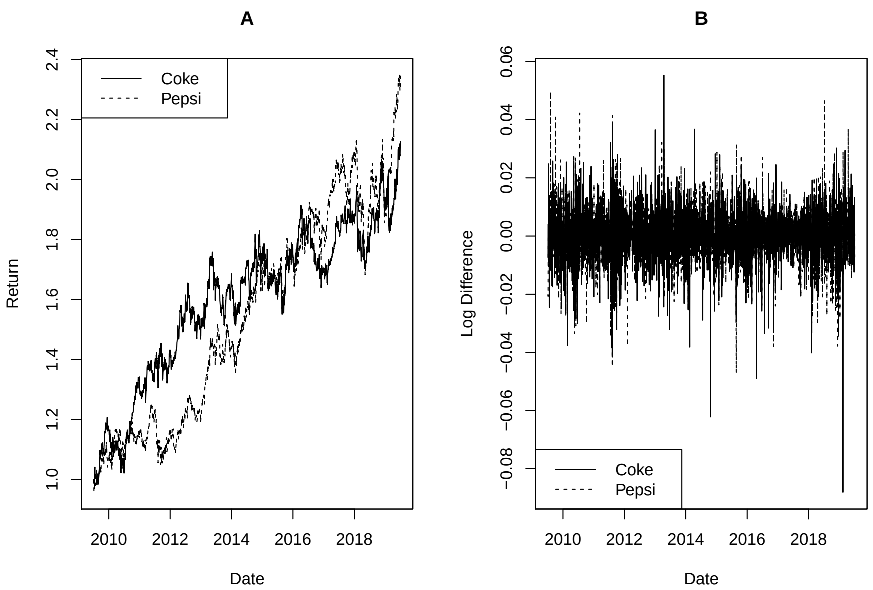

Figure 3: Plots of Returns and Log-differences of Stock Prices of Coca-cola and Pepsi

Table 2: Changes in the stock prices with the corresponding probabilities

Change $[\Delta S_1, \Delta S_2]^T$	Number of Occurrence	Probability
$[[1, 0]^T$	3	0.004762
$[-1, 0]^T$	3	0.004762
$[0, 1]^T$	1	0.001587
$[0, -1]^T$	2	0.003175
$[1, 1]^T$	264	0.419048
$[1, -1]^T$	70	0.111111
$[-1, 1]^T$	72	0.114286
$[-1, -1]^T$	215	0.341269
$[0, 0]^T$	0	0.000000
Total	630	1.000000

thus: The drift coefficient is obtained from Table 2 as

$$\mu(t, S_1, S_2) = \begin{bmatrix} (b_1 - d_1)S_1 + (m_{22} + m_{21} - m_{12} - m_{11})S_1S_2 \\ (b_2 - d_2)S_2 + (m_{22} - m_{21} + m_{12} + m_{11})S_1S_2 \end{bmatrix}$$

$$(b_1 - d_1)S_1 = 0.004762 - 0.004762 = 0.000000$$

$$(m_{22} + m_{21} - m_{12} - m_{11})S_1S_2 = 0.419048 + 0.111111 - 0.114286 - 0.341269 = 0.074603$$

It follows also from Table that,

$$(b_2 - d_2)S_2 = -0.001587$$

$$(m_{22} - m_{21} + m_{12} - m_{11})S_1S_2 = 0.080952$$

Therefore, we have

$$\mu(t, S_1, S_2) = \begin{bmatrix} 0.000000 + 0.074603 \\ -0.001587 + 0.080952 \end{bmatrix} = \begin{bmatrix} 0.074603 \\ 0.079365 \end{bmatrix}$$

But $c_1 = (b_1+d_1)S_1 = 0.009524$, $c_2 = (m_{22}+m_{11})S_1S_2 = 0.760317$, $c_3 = (m_{21}+m_{12})S_1S_2 = 0.225396$ and $c_4 = (b_2 + d_2)S_2 = 0.004762$. With $w = \sqrt{(c_1 + c_2 + c_3)(c_2 + c_3 + c_4)(c_2 - c_3)^2} = 0.836433$ and $d = \sqrt{c_1 + 2c_2 + 2c_3 + c_4 + 2w} = 1.912741$. Hence, the volatility can be obtained thus:

$$B(t, S_1, S_2) = \frac{1}{d} \begin{bmatrix} c_1 + c_2 + c_3 + w & c_2 - c_3 \\ c_2 - c_3 & c_2 + c_3 + c_4 + w \end{bmatrix}$$

Substituting the values of the variables, gives

$$B(t, S_1, S_2) = \begin{bmatrix} 0.957615 \; 0.279662 \\ 0.279662 \; 0.955125 \end{bmatrix}$$

Therefore the resulting stochastic differential equation becomes

$$dS(t) = \mu(t, S_1, S_2)dt + B(t, S_1, S_2)dW(t)$$

$$dS(t) = \begin{bmatrix} 0.074603 \\ 0.079365 \end{bmatrix} dt + \begin{bmatrix} 0.957615 \; 0.279662 \\ 0.279662 \; 0.955125 \end{bmatrix} dW(t) \tag{8}$$

The resulting SDE equation (8) is the SDE model for the two stocks, with the estimated drift and volatility parameters. The simulation to the SDE is executed using the multi-dimensional Euler-Maruyama scheme for SDEs and implemented using the R package called ”Sim.DiffProc” by Boukhetala and Guidoum (2018) and the ”yuima” package by Brouste et al. (2014). Simulation here is not data simulation; it is the modelling of the trajectory by a numerical scheme (Euler-Maruyama) of the given the SDE. For more on Euler-Maruyama approximation technique refer to Kloeden and Platen (1995) or Platen and Bruti-Liberati (2010).

3.3 *Diffusion process for the stock prices model*

The diffusion process of the SDE in equation (8) which is its solution is obtained using the Euler-Maruyama method implemented using the R packages mentioned in the previous section. The simulation trajectory of the process solution of the SDE gives approximation to the solutions of the multidimensional SDE with Wiener process. The solutions to the multidimensional SDEs are usually not available in explicit analytical form, are evaluated using Numerical Methods which are usually based on discrete approximations of the continuous solution to a stochastic differential equation (Boukhetala and Guidoum, 2018). The Monte-Carlo statistics of the simulation for the SDE are given in the table below.

Table 3: Monte-Carlo statistics for S_1 and S_2

Statistic	S_1(Coca-cola)	S_2(Pepsi)
Mean	-0.01355	-0.05066
Variance	0.9986	1.01174
Median	-0.0026	0.00033
Skewness	-0.0838	0.0410
Kurtosis	3.116	3.003
Coef-variation	-73.095	-19.928

The quadratic covariation estimates between the two states of the stock price system, S_1 and S_2 from the diffusion process are obtained as given below

$$Cov.matrix = \begin{bmatrix} 1.012 \; 0.545 \\ 0.545 \; 0.999 \end{bmatrix}$$

$$Corr.matrix = \begin{bmatrix} 1.000 & 0.542 \\ 0.542 & 1.000 \end{bmatrix}$$

The correlation matrix from the diffusion process reveals the association of the two stock prices system as evident in the plot of the returns of both stocks in Figure 3. The basis is that the same environmental and economic factors influence both stocks and their overall movements are in the same direction as seen in the diffusion plot in Figure 6.

The marginal densities of the two stocks simulated are shown in Figure 4 below to be approximately normally distributed; this shows the SDE model is close for both stocks, which are in the same industry and hence suitable stock price model. The joint densities of the two stocks simulated shows the perspective plot

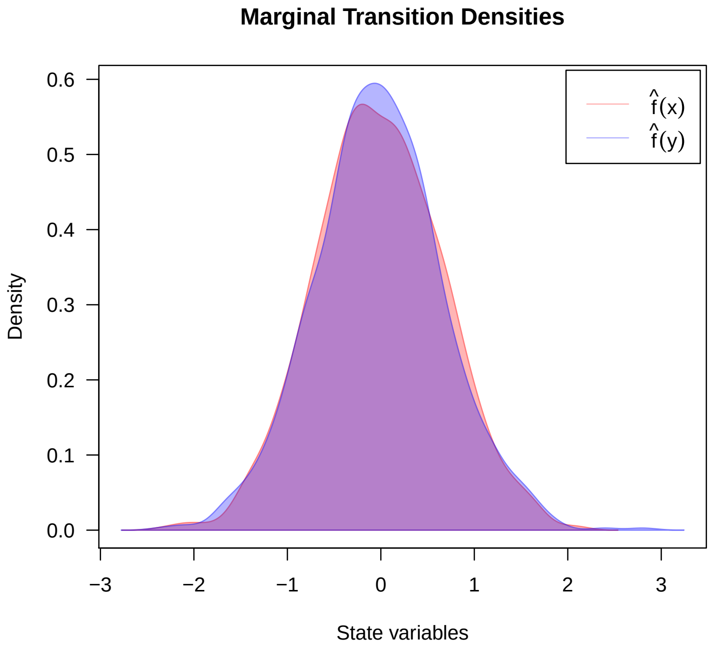

Figure 4: Plot of Marginal Densities of the Diffusion Process for both Stocks

in Figure 5 below of how both stocks paths vary close to zero. This further shows how stocks change in short period of time as a result of external market factors and other human and natural factors influencing the markets. Figure 6 below shows the plot of the diffusion process of the SDE for both stocks. The state of the system represented by S_1 describes Coke stock, while S_2 describes Pepsi stock. The accumulated pattern of the simulation of the two stocks which is the cumulative sum of movement of the two stocks can be seen from the plot. The pattern indicates a Brownian motion as the path is seen to be unpredictable or chaotic for a long period of time. Comparing the plot of Figure 6 with the plot of Figure 2, it can be seen that the diffusion process models the stock prices fairly well as both plots for each stock show consistent movements. It can be seen that the diffusion process of the SDE for both stocks models the movement of the stocks.

4. Conclusion

Nonlinear dynamical systems describing changes in stock prices over time may appear chaotic and are difficult to solve. The systems can commonly be approximated by linear equations (linearization) using a differential equation. Unlike deterministic models such as ordinary differential equations, which have a unique solution for each appropriate initial condition, SDEs have solutions that are continuous-time stochastic processes. The Diffusion process for the SDE was simulated using the multi-dimensional Euler-Maruyama scheme for SDEs implemented in R statistical packages. It was observed that the diffusion process of the SDE for both stocks modeled the movement of the actual data obtained. The diffusion process of the SDE for both stocks on the actual data obtained showed that the diffusion process modeled the Markov process of the stock prices. This is a process that used forward Kolmogorov equation for the continuous-time stochastic process which approximates the partial differential equation and corresponds to a Diffusion Process having

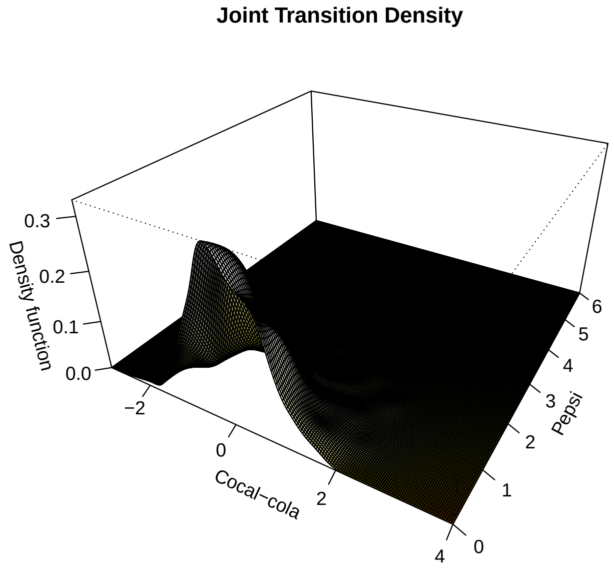

Figure 5: Perspective plot of joint density of diffusion process for both Stocks

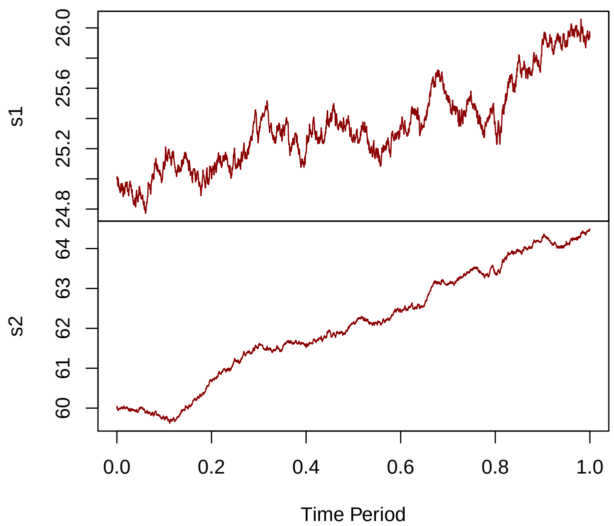

Figure 6: Plot of the Diffusion Process of both states of the System

the stochastic differential equation. The study showed the building of stochastic differential model using continuous-time Markov diffusion processes with a structural approach using available discrete time realisations of the stochastic process with application to stock prices. The modeling procedure can be extended to more than two stock prices for the purpose of financial portfolio analyses and management for decision making and competitive advantage as well as stochastic differential equations other than the Ito type, such as Stratonovich stochastic differential equation.

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