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On the Stationarity of Multivariate Time Series

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Abstract. The purpose of this work is to investigate the effect of non-stationary process in the multivariate time series. The investigation was carried out using three vector series X_{1t}, X_{2t} and X_{3t} representing the average, the urban and the rural consumer price indices respectively in Nigeria. The data were collected from CBN Statistical Bulletin from 1995-2018. Of the three vector series, X_{2t} (urban price index) was conditioned to be in the state of disequilibrium (non-stationary state). The cross-autocorrelation matrix of the three vector series up to the second lag produced nine sub-cross-autocorrelation matrices. The sub-matrices are made up of negative and positive cross-autocorrelations. The positive definiteness property was investigated for each sub-autocorrelation matrix. The joint processes with positive cross-correlations met the stationarity condition, while processes with negative cross-correlations violated the condition. The violation of the stationarity condition was as a result of the unstable urban consumer price index. The investigation revealed that when a state is unstable in the multivariate time series, the stability of the multivariate process is affected by way of partial stationarity.

Keywords: Non-stationarity, positive-definiteness, cross-autocovariance and cross-autocorrelation.

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1. Introduction

The test for stationarity of a process is a common phenomenon in time series analysis. As postulated by different authors, stability of a time series process implies stationarity as the process is said to be in equilibrium state (Lutkepohl, 2005). Stationarity of a time series process has the property that the mean, variance and autocorrelation structure do not change over time. If the time series is not stationary, it can be transformed to become stationary with some time series operators or techniques. These may include: application of the difference operators $\bigtriangledown^d = (1 - B)^d$, or taking the logarithm or square root of the series to stabilize the variance for non-constant variance, etc. For instance, for a non-seasonal time series process, if $d = 1$, $\bigtriangledown X_t = X_t - X_{t-1}$ or $T_t = a + bt$. The initial process taken to visualize the behaviour of a process is the time graph plotted to display some hidden features that characterise the series. The time plot exhibits certain attributes that could not be seen by raw data inspection, and these may include trend, seasonality or both. This justifies the need for the adoption of an operator or statistical technique to ensure that the equilibrium state of the process is attained. Indisputably, most of the original economic and financial time series often show upward trend (increase in series); hence, giving room for difference or trend analysis to attain stability.

In univariate time series, the unit root test is a frequently used statistical procedure for testing the stationarity. In conducting the test, if a given time series has a unit root; it implies the time series is not stationary. The test is carried out to either agree with the null hypothesis that the time series is not stationary or reject for the alternative hypothesis that the series is stationary (Kiyang and Shahabi, 2005). In carrying out a unit root test, a simple regression model is estimated.

$$X_t = \rho X_{t-1} + \epsilon_t \tag{1}$$

where, X_t is an observation at time t, and ϵ_t is a sequence of independent normal random variables with mean 0 and variance σ^2. The unit root test has the hypothesis that $H_0 : \rho = 1$ against $H_1 : \rho < 1$. If H_0 is accepted, it means X_t is not stationary (Dickey and Fuller, 1979). The modification of the test is Augmented

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Dickey-Fuller test, which involves an autoregressive model that contains the trend and drift (Patrick, 2020). In the multivariate case, Co-integration test is one of the popular tests adopted to test for stationarity of a set of time series. The test considers investigating the individual univariate time series processes making up the multivariate and checking if all the time processes are stationary. When checking for stationarity of multivariate time series, consideration is also given to the fact that some of the time series making up the multivariate time series are stationary and some are not stationary (Kiyang and Shahabi, 2005).

The aim of the study is to investigate stationarity of multivariate time series in the midst of stable and unstable processes so as to ascertain if an unstable state has effect on the whole multivariate time series. Assessing the structure of the autocovariances and autocorrelations is very prominent in investigating the stationarity of time series process (see Box and Jenkins, 1976; Kendall and Ord, 1990; Gujarati and Porter, 2009). Apart from the unit root test, which involves regression, autocovariance/autocorrelation structure of the univariate time series is also investigated to find out if the autocovariances or autocorrelations are from a stationary process. As a condition, positive definiteness or semi-positive definiteness property is investigated to ascertain the stationarity of the autocovariance structure with the autocorrelation matrix at different time lags. This procedure is extended to multivariate time series (Engle and Kroner, 1995). Patrick (2020) considered a power envelop properties for tests against both stationary and explosive alternatives. The study was conducted to check the effect of trend on the non-stationary time series. Other stationarity tests in umivariate time series are Brendan (2018) and Helmut et al. (2019). In multivariate time series, the n-dimensional cross-autcovariance or cross-autocorrelation matrix composes of sub-matrices of individual vector processes with distributed lags. For a cross-autocorrelation matrix to be positive definite, it is a justifiable assumption that the individual sub-autocorrelation matrices meet the positive definiteness condition. That means, their determinants and principal minors have positive values. Bollersler et al. (1998) considered $k = 2$ ARCH(1) process and symmetric positive definite matrix of cross-covariances. Kiyang and Shahabi (2005) investigated stationarity of multivariate time series for correlation-based data analysis using a method called Corona and Eros. Corona is a supervised features subset selection technique for multivariate time series data set. Each Multivariate time series data is firstly represented correlation coefficients, which are subsequently transformed into vector. Data are transformed using a Support Vector Machine (SVM), and each variable that contributes the least to the separating hyperplane is recursively eliminated. Eros is a measure for multivariate time series data set that is based on principal component analysis (PCA). Eros compute the similarity of two multivariate time series data by measuring two corresponding principal components using the aggregated eigenvalues as weight. Jensten and Rao (2015) adopted the Discrete Fourier Transform as a tool to test for second order stationarity of multivariate time series. Ruprecht and Philip (2016) proposed a non-parametric procedure for validating the assumption of stationarity in multivariate locally stationary time series models. A boostrap-assisted test based on a kolomogorov-smirnov type statistic. This is contrary to parametric based test for detecting stationarity in time series. Furtherance to the aforementioned contributions, positive definiteness property as a tool to investigate multivariate stationarity is the major concern in this study.

In addition to the investigation of the assumption of positive definiteness of the n-dimensional cross-autocovariance matrix, it becomes more revealing to investigate the positive definiteness of the sub-autocovariance matrices of individual vectors as the components of the cross-covariance and cross-correlation matrix. This is considered as the first step of verifying stationarity property before the larger cross-covariance matrix is investigated. Each univariate autocovariance structure making up the cross-autocovariance matrix is verified; and the investigation is concluded in the cross-autocovariance or cross-autocorrelation matrix. This paper uses both the sub-covariance/correlation matrices and cross-autocovraiance/autocorrelation matrix to ascertain stationarity of the multivariate time series process.

2. Materials and method

Consider the cross-covariances of the vector processes

$$\gamma_{1t+k,jt+l} = \begin{bmatrix} \gamma_{1t+k,1t+l} & \gamma_{1t+k,2t+l} & \gamma_{1t+k,3t+l} & \cdots & \gamma_{1t+k,nt+l} \\ \gamma_{2t+k,1t+l} & \gamma_{2t+k,2t+l} & \gamma_{2t+k,3t+l} & \cdots & \gamma_{2t+k,nt+l} \\ \gamma_{3t+k,1t+l} & \gamma_{3t+k,2t+l} & \gamma_{3t+k,3t+l} & \cdots & \gamma_{3t+k,nt+l} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{mt+k,1t+l} & \gamma_{mt+k,2t+l} & \gamma_{mt+k,3t+l} & \cdots & \gamma_{mt+k,nt+l} \end{bmatrix} \tag{2}$$

where $i = 1, \ldots, m$; $j = 1, \ldots, n$; $k = 1, \ldots, r$; $l = 1, \ldots, s$ Equation (2) is the cross-covariance matrix. The above matrix is symmetric and the cross-covariance of each i and j for r=s can be expressed in a triangular form as shown from equations (3) to (6).

$$\gamma_{1t+k,1t} = \gamma_{1t,1t+l} \begin{bmatrix} \gamma_{1t,1t} & & & & \\ \gamma_{1t+1,1t} & \gamma_{1t+1,1t+1} & & & \\ \gamma_{1t+2,1t} & \gamma_{1t+2,1t+1} & \gamma_{1t+2,1t+2} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{1t+r,1t} & \gamma_{1t+r,1t+1} & \gamma_{1t+r,1t+2} & \cdots & \gamma_{1t+r,1t+s} \end{bmatrix} \quad (3)$$

$$\gamma_{2t+k,2t} = \gamma_{2t,2t+l} \begin{bmatrix} \gamma_{2t,2t} & & & & \\ \gamma_{2t+1,2t} & \gamma_{2t+1,2t+1} & & & \\ \gamma_{2t+2,2t} & \gamma_{2t+2,2t+1} & \gamma_{2t+2,2t+2} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{2t+r,2t} & \gamma_{2t+r,2t+1} & \gamma_{2t+r,2t+2} & \cdots & \gamma_{2t+r,2t+s} \end{bmatrix} \quad (4)$$

$$\gamma_{3t+k,3t} = \gamma_{3t,3t+l} \begin{bmatrix} \gamma_{3t,3t} & & & & \\ \gamma_{3t+1,3t} & \gamma_{3t+1,3t+1} & & & \\ \gamma_{3t+2,3t} & \gamma_{3t+2,3t+1} & \gamma_{3t+2,3t+2} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{3t+r,3t} & \gamma_{3t+r,3t+1} & \gamma_{3t+r,3t+2} & \cdots & \gamma_{3t+r,3t+s} \end{bmatrix} \quad (5)$$

$$\vdots$$

$$\gamma_{mt+k,mt} = \gamma_{mt,mt+l} \begin{bmatrix} \gamma_{mt,mt} & & & & \\ \gamma_{mt+1,mt} & \gamma_{mt+1,mt+1} & & & \\ \gamma_{mt+2,mt} & \gamma_{mt+2,mt+1} & \gamma_{mt+2,mt+2} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{mt+r,mt} & \gamma_{mt+r,mt+1} & \gamma_{mt+r,mt+2} & \cdots & \gamma_{mt+r,mt+s} \end{bmatrix} \quad (6)$$

Equations (3), (4), (5) and (6) are covariances of X_{it+k} and X_{jt} or X_{it} and X_{jt+l} ($i = j$ at $k = 0, 1, \ldots, r$ or $l = 0, 1, \ldots, s$). For $i = j$ and $k = l$, $r = s$, the covariances of X_{it+k} and X_{jt+l} denoted by $\gamma_{it+k,jt+l}$ are variances and these represented in the principal diagonal of the covariance matrices. The lower triangular elements, denoted by $\gamma_{it+k,jt+l}$ are the covariance matrices.

$$\gamma_{1t+k,2t+l} = \begin{bmatrix} \gamma_{1t,2t} & \gamma_{1t,2t+1} & \gamma_{1t,2t+2} & \cdots & \gamma_{1t,2t+s} \\ \gamma_{1t+1,2t} & \gamma_{1t+1,2t+1} & \gamma_{1t+1,2t+2} & \cdots & \gamma_{1t+1,2t+s} \\ \gamma_{1t+2,2t} & \gamma_{1t+2,2t+1} & \gamma_{1t+2,2t+2} & \cdots & \gamma_{1t+2,2t+s} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{1t+r,2t} & \gamma_{1t+r,2t+1} & \gamma_{1t+r,2t+2} & \cdots & \gamma_{1t+r,2t+s} \end{bmatrix} \quad (7)$$

$$\gamma_{1t+k,3t+l} = \begin{bmatrix} \gamma_{1t,3t} & \gamma_{1t,3t+1} & \gamma_{1t,3t+2} & \cdots & \gamma_{1t,3t+s} \\ \gamma_{1t+1,3t} & \gamma_{1t+1,3t+1} & \gamma_{1t+1,3t+2} & \cdots & \gamma_{1t+1,3t+s} \\ \gamma_{1t+2,3t} & \gamma_{1t+2,3t+1} & \gamma_{1t+2,3t+2} & \cdots & \gamma_{1t+2,3t+s} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{1t+r,3t} & \gamma_{1t+r,3t+1} & \gamma_{1t+r,3t+2} & \cdots & \gamma_{1t+r,3t+s} \end{bmatrix} \quad (8)$$

$$\vdots$$

$$\gamma_{1t+k,nt+l} = \begin{bmatrix} \gamma_{1t,nt} & \gamma_{1t,nt+1} & \gamma_{1t,nt+2} & \cdots & \gamma_{1t,nt+s} \\ \gamma_{1t+1,nt} & \gamma_{1t+1,nt+1} & \gamma_{1t+1,nt+2} & \cdots & \gamma_{1t+1,nt+s} \\ \gamma_{1t+2,nt} & \gamma_{1t+2,nt+1} & \gamma_{1t+2,nt+2} & \cdots & \gamma_{1t+2,nt+s} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{1t+r,nt} & \gamma_{1t+r,nt+1} & \gamma_{1t+r,nt+2} & \cdots & \gamma_{1t+r,nt+s} \end{bmatrix} \tag{9}$$

Other matrices include

$$\gamma_{2t+k,3t+l}, \gamma_{2t+k,nt+l}, \gamma_{3t+k,nt+l}, \cdots, \gamma_{mt+k,nt+l} \tag{10}$$

Equations (7), (8), (9) and (10) are cross-covariances of X_{it+k} and X_{jt+l} ($i \neq j$ at $k = 0, 1, \ldots, r$ and $l = 0, 1 \ldots, s$). The matrices (7), (8), (9) and (10) are not symmetrical and the diagonal elements are not variances. The elements of the matrices are constituted by $\gamma_{it+k,jt+l}$, for $i \neq j, k = l$ and $k \neq l$.

2.1 *Auto-correlations and cross-auto-correlations*

Autocorrelations measure the correlations between the same variable at different lags (X_{it+k} and X_{jt} or X_{it} and X_{jt+l} ; $i = j$ at $k = 0, 1, \ldots, r$ or $l = 0, 1, \ldots, s$), while cross-autocorrelations measure correlations between different variables at different lags (X_{it} and X_{jt}, $i \neq j$ at $k = 0, 1, \ldots, r$ and $l = 0, 1, \ldots, s$).

In a univariate time series, autocorrelation is given as

$$\rho_{s,t} = \frac{E(X_t - \mu)(X_s - \mu)}{(\sigma_t \sigma_s)} \tag{11}$$

where E is the expected value operator,X_t, X_s are two processes with standard deviations σ_t, σ_s and mean μ.

Usoro (2015) obtained vector cross-correlation as

$$\rho_{it+k,jt+l} = \frac{\gamma_{it+k,jt+l}}{\sqrt{(}\gamma_{it}\gamma_{jt})} \tag{12}$$

The above autocorrelation is obtained by dividing the square root of the two variances γ_{it} and γ_{jt}. Hence, equations (3), (4), (5), (6), (7), (8), (9) and (10) are divided by the square root of the product variances to become

$$\rho_{1t+k,1t} = \rho_{1t,1t+l} \begin{bmatrix} \rho_{1t,1t} & & & & \\ \rho_{1t+1,1t} & \rho_{1t+1,1t+1} & & & \\ \rho_{1t+2,1t} & \rho_{1t+2,1t+1} & \rho_{1t+2,1t+2} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{1t+r,1t} & \rho_{1t+r,1t+1} & \rho_{1t+r,1t+2} & \cdots & \rho_{1t+r,1t+s} \end{bmatrix} \tag{13}$$

$$\rho_{2t+k,2t} = \rho_{2t,2t+l} \begin{bmatrix} \rho_{2t,2t} & & & & \\ \rho_{2t+1,2t} & \rho_{2t+1,2t+1} & & & \\ \rho_{2t+2,2t} & \rho_{2t+2,2t+1} & \rho_{2t+2,2t+2} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{2t+r,2t} & \rho_{2t+r,2t+1} & \rho_{2t+r,2t+2} & \cdots & \rho_{2t+r,2t+s} \end{bmatrix} \tag{14}$$

$$\rho_{3t+k,3t} = \rho_{3t,3t+l} \begin{bmatrix} \rho_{3t,3t} & & & & \\ \rho_{3t+1,3t} & \rho_{3t+1,3t+1} & & & \\ \rho_{3t+2,3t} & \rho_{3t+2,3t+1} & \rho_{3t+2,3t+2} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{3t+r,3t} & \rho_{3t+r,3t+1} & \rho_{3t+r,3t+2} & \cdots & \rho_{3t+r,3t+s} \end{bmatrix} \tag{15}$$

$$\vdots$$

$$\rho_{mt+k,mt} = \rho_{mt,mt+l} \begin{bmatrix} \rho_{mt,mt} & & & & \\ \rho_{mt+1,mt} & \rho_{mt+1,mt+1} & & & \\ \rho_{mt+2,mt} & \rho_{mt+2,mt+1} & \rho_{mt+2,mt+2} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{mt+r,mt} & \rho_{mt+r,mt+1} & \rho_{mt+r,mt+2} & \cdots & \rho_{mt+r,mt+s} \end{bmatrix} \tag{16}$$

Equations (13), (14), (15) and (16) are autocorrelations of X_{it+k} and X_{jt+l} $(i = j$ at $k = 0, 1, \ldots, r$ or $l = 0, 1, \ldots, s)$. The principal diagonal elements have correlations of same vectors with $\rho_{it+k,jt+l(i=j,k=l))} = 1$.

$$\rho_{1t+k,2t+l} = \begin{bmatrix} \rho_{1t,2t} & \rho_{1t,2t+1} & \rho_{1t,2t+2} & \cdots & \rho_{1t,2t+s} \\ \rho_{1t+1,2t} & \rho_{1t+1,2t+1} & \rho_{1t+1,2t+2} & \cdots & \rho_{1t+1,2t+s} \\ \rho_{1t+2,2t} & \rho_{1t+2,2t+1} & \rho_{1t+2,2t+2} & \cdots & \rho_{1t+2,2t+s} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{1t+r,2t} & \rho_{1t+r,2t+1} & \rho_{1t+r,2t+2} & \cdots & \rho_{1t+r,2t+s} \end{bmatrix} \tag{17}$$

$$\rho_{1t+k,3t+l} = \begin{bmatrix} \rho_{1t,3t} & \rho_{1t,3t+1} & \rho_{1t,3t+2} & \cdots & \rho_{1t,3t+s} \\ \rho_{1t+1,3t} & \rho_{1t+1,3t+1} & \rho_{1t+1,3t+2} & \cdots & \rho_{1t+1,3t+s} \\ \rho_{1t+2,3t} & \rho_{1t+2,3t+1} & \rho_{1t+2,3t+2} & \cdots & \rho_{1t+2,3t+s} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{1t+r,3t} & \rho_{1t+r,3t+1} & \rho_{1t+r,3t+2} & \cdots & \rho_{1t+r,3t+s} \end{bmatrix} \tag{18}$$

$$\vdots$$

$$\rho_{1t+k,nt+l} = \begin{bmatrix} \rho_{1t,nt} & \rho_{1t,nt+1} & \rho_{1t,nt+2} & \cdots & \rho_{1t,nt+s} \\ \rho_{1t+1,nt} & \rho_{1t+1,nt+1} & \rho_{1t+1,nt+2} & \cdots & \rho_{1t+1,nt+s} \\ \rho_{1t+2,nt} & \rho_{1t+2,nt+1} & \rho_{1t+2,nt+2} & \cdots & \rho_{1t+2,nt+s} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{1t+r,nt} & \rho_{1t+r,nt+1} & \rho_{1t+r,nt+2} & \cdots & rho_{1t+r,nt+s} \end{bmatrix} \tag{19}$$

Other matrices include

$$\rho_{2t+k,3t+l},\ \rho_{2t+k,nt+l},\ \rho_{3t+k,nt+l},\ \cdots,\ \rho_{mt+k,nt+l} \tag{20}$$

Equations (17), (18), (19) and (20) are cross-autocorrelations of X_{it+k} and X_{jt+l} $(i \neq j$ at $k = 0, 1, \ldots, r$ and $l = 0, 1, \ldots, s)$. The upper and the lower elements of the matrices are constituted by $\gamma_{it+k,jt+l}$, for $i \neq j$ and $k \neq l$. The principal diagonal elements are cross-autocorrelations with $\rho_{it+k,jt+l(i\neq j,k=l))} \neq 1$.

2.2 *Positive definitness of auto-correlations and cross-auto-correlation matrices*

The positive definiteness of the autocorrelation matrix requires that all the principal minors of the autocorrelation matrix are greater than zero. Also the determinant of the autocorrelation matrix is greater than zero (Box and Jenkins, 1976). Given the auto-correlation and cross-autocorrelation matrix of stationary processes

$$\rho_{1t+k,jt+l} = \begin{bmatrix} \rho_{1t+k,1t+l} & \rho_{1t+k,2t+l} & \rho_{1t+k,3t+l} & \cdots & \rho_{1t+k,nt+l} \\ \rho_{2t+k,1t+l} & \rho_{2t+k,2t+l} & \rho_{2t+k,3t+l} & \cdots & \rho_{2t+k,nt+l} \\ \rho_{3t+k,1t+l} & \rho_{3t+k,2t+l} & \rho_{3t+k,3t+l} & \cdots & \rho_{3t+k,nt+l} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{mt+k,1t+l} & \rho_{mt+k,2t+l} & \rho_{mt+k,3t+l} & \cdots & \rho_{mt+k,nt+l} \end{bmatrix} \tag{21}$$

The positive definiteness of $\rho_{it+k,jt+l}$ requires that:

(i) The minors of $\rho_{1t+k,1t+l}$,$\rho_{2t+k,2t+l}$, $\rho_{3t+k,3t+l}$ and $\rho_{mt+k,nt+l}$ are greater than zero
(ii) The determinant of $\rho_{it+k,jt+l} > 0$

3. Results and discussion

In this section, cross-autocorrelations involving stationary (log return) average consumer price index, (log return) rural consumer price index and non-stationary urban consumer price index of Nigeria are estimated. The data source is from CBN Statistical Bulletin from 1995 to 2018 ***(httpp://statistics.cbn.gov.ng/cbn-onlinestats/DataBrowser.aspx)***. The estimates are in the following autocorrelation and cross-autocorrelation matrix.

$$\rho_{1t+k,jt+l} = \begin{bmatrix} 1.000 & 0.258 & 0.250 & -0.088 & -0.082 & -0.078 & 0.922 & 0.213 & 0.213 \\ 0.258 & 1.000 & 0.257 & -0.089 & -0.085 & -0.078 & 0.204 & 0.922 & 0.213 \\ 0.259 & 0.258 & 1.000 & -0.091 & -0.086 & -0.081 & 0.228 & 0.204 & 0.992 \\ -0.088 & -0.089 & -0.091 & 1.000 & 1.000 & 0.999 & -0.113 & -0.114 & -0.116 \\ -0.082 & -0.085 & -0.086 & 1.000 & 1.000 & 1.000 & -0.107 & -0.110 & -0.111 \\ -0.078 & -0.078 & -0.081 & 0.999 & 1.000 & 1.000 & -0.104 & -0.104 & -0.107 \\ 0.922 & 0.204 & 0.228 & -0.113 & -0.107 & -0.104 & 1.000 & 0.175 & 0.192 \\ 0.213 & 0.922 & 0.204 & -0.114 & -0.110 & -0.104 & 0.175 & 1.000 & 0.174 \\ 0.213 & 0.213 & 0.922 & -0.116 & -0.111 & -0.107 & 0.192 & 0.174 & 1.000 \end{bmatrix} \tag{22}$$

where $i = 1, 2, 3; j = 1, 2, 3; k = 0, 1, 2; l = 0, 1, 2$. The sub-matrices which make up the components of $\rho_{it+k,jt+l}$ (22) are provided below

Case 1: Matrix of autocorrelations of $X_{1t,1t+k(k=0,1,2)}$

$$\rho_{1t+k,1t+l} = \begin{bmatrix} 1.000 & 0.258 & 0.250 \\ 0.258 & 1.000 & 0.257 \\ 0.259 & 0.258 & 1.000 \end{bmatrix}$$

The principal minors of $\rho_{1t+k,1t+l}$ are: $\rho_{1t+k,1t+l(k=1,2;l=1,2)} = 0.93395$, $\rho_{1t+k,1t+l(k=0,2;l=0,2)} = 0.9375$, $\rho_{1t+k,1t+l(k=0,1;l=0,1)} = 0.93334$, $Det(\rho_{1t+k,1t+l}) = 0.8380$. The principal minors and determinant of $\rho_{1t+k,1t+l} > 0$. Positive definiteness condition is met.

Case 2: Matrix of cross-autocorrelations of $X_{1t+k,2t+l(k,l=0,1,2)}$

$$\rho_{1t+k,2t+l} = \begin{bmatrix} -0.088 & -0.082 & -0.078 \\ -0.089 & -0.085 & -0.078 \\ -0.091 & -0.086 & -0.081 \end{bmatrix}$$

The principal minors of $\rho_{1t+k,2t+l}$ are: $\rho_{1t+k,2t+l(k=1,2;l=1,2)} = 0.000177$, $\rho_{1t+k,2t+l(k=0,2;l=0,2)} = 0.000382$, $\rho_{1t+k,2t+l(k=0,1;l=0,1)} = 0.000182$, $Det(\rho_{1t+k,2t+l}) = -0.000000156$. The principal minors of $\rho_{1t+k,2t+l} > 0$ and determinant of $\rho_{1t+k,2t+l} < 0$. Also, The principal minors of $\rho_{2t+k,1t+l} > 0$ and determinant of $\rho_{2t+k,1t+l} < 0$. Positive definiteness condition is not met.

Case 3: Matrix of cross-autocorrelations of $X_{1t+k,3t+l(k,l=0,1,2)}$

$$\rho_{1t+k,3t+l} = \begin{bmatrix} 0.922 & 0.213 & 0.213 \\ 0.204 & 0.922 & 0.213 \\ 0.228 & 0.204 & 0.992 \end{bmatrix}$$

The principal minors of $\rho_{1t+k,3t+l}$ are: $\rho_{1t+k,3t+l(k=1,2;l=1,2)} = 0.806632$, $\rho_{1t+k,3t+l(k=0,2;l=0,2)} = 0.80152$, $\rho_{1t+k,3t+l(k=0,1;l=0,1)} = 0.806632$, $Det(\rho_{1t+k,3t+l}) = 0.678084292$. The principal minors of $\rho_{1t+k,3t+l} > 0$ and determinant of $\rho_{1t+k,3t+l} > 0$. Also, The principal minors of $\rho_{3t+k,1t+l} > 0$ and determinant of $\rho_{3t+k,1t+l} > 0$. Positive definiteness condition is met.

Case 4: Matrix of autocorrelations of $X_{2t+k,2t+l(k,l=0,1,2)}$

$$\rho_{2t+k,2t+l} = \begin{bmatrix} 1.000 & 1.000 & 0.999 \\ 1.000 & 1.000 & 1.000 \\ 0.999 & 1.000 & 1.000 \end{bmatrix}$$

The principal minors of $\rho_{2t+k,2t+l}$ are: $\rho_{2t+k,2t+l(k=1,2;l=1,2)} = 0$, $\rho_{2t+k,2t+l(k=0,2;l=0,2)} = 0.001999$, $\rho_{2t+k,2t+l(k=0,1;l=0,1)} = 0$, $Det(\rho_{2t+k,2t+l}) = -0.000001$. The principal minors of $\rho_{2t+k,2t+l} > 0$ and determinant of $\rho_{2t+k,1t+l} < 0$. Positive definiteness condition is not met.

Case 5: Matrix of cross-autocorrelations of $X_{2t+k,3t+l(k,l=0,1,2)}$

$$\rho_{2t+k,3t+l} = \begin{bmatrix} -0.113 & -0.114 & -0.116 \\ -0.107 & -0.110 & -0.111 \\ -0.104 & -0.104 & -0.107 \end{bmatrix}$$

The principal minors of $\rho_{2t+k,3t+l}$ are: $\rho_{2t+k,3t+l(k=1,2;l=1,2)} = 0.000226$, $\rho_{2t+k,3t+l(k=0,2;l=0,2)} = -0.125064$, $\rho_{2t+k,3t+l(k=0,1;l=0,1)} = 0.000232$, $Det(\rho_{2t+k,3t+l}) = -0.000000076$. The principal minors of $\rho_{2t+k,3t+l} < 0$ and determinant of $\rho_{2t+k,3t+l} < 0$. Also, The principal minors of $\rho_{3t+k,2t+l} < 0$ and determinant of $\rho_{3t+k,2t+l} < 0$. Positive definiteness condition is not met.

Case 6: Matrix of autocorrelations of $X_{3t+k,3t+l(k,l=0,1,2)}$

$$\rho_{3t+k,3t+l} = \begin{bmatrix} 1.000 & 0.175 & 0.192 \\ 0.175 & 1.000 & 0.174 \\ 0.192 & 0.174 & 1.000 \end{bmatrix}$$

The principal minors of $\rho_{3t+k,3t+l}$ are: $\rho_{3t+k,3t+l(k=1,2;l=1,2)} = 0.969724$, $\rho_{3t+k,3t+l(k=0,2;l=0,2)} = 0.903136$, $\rho_{3t+k,3t+l(k=0,1;l=0,1)} = 0.969375$, $Det(\rho_{3t+k,3t+l}) = 0.9139278$. The principal minors of determinant of $\rho_{3t+k,3t+l} > 0$. Positive definiteness condition is met.

From (22), the principal minors and determinants of some submatrices are greater than zero and some are less than zero. Therefore, positive definiteness condition not met.

The principal minors and determinant of the autocorrelation matrix are considered. The first row and column of the 9x9 autocorrelation matrix are $\rho_{it+k,jt+l[(i=1;k=0),(j=1,2,3;l=0,1,2)])}$ and $\rho_{it+k,jt+l[(i=1,2,3;k=0,1,2),(j=1;l=0)]}$. Since (22) is symmetric, the crossed elements of the first row and column can be written as: $\rho_{it+k,jt+l[(i=1;k=0),(j=1,2,3;l=0,1,2)]}$ and $\rho^{I}_{it+k,jt+l[(i=1;k=0),(j=1,2,3;l=0,1,2))}$ with the respective minor

$\rho_{1t+k,jt+l[(i=1;k=1,2;i=2,3,k=0,1,2),(j=1,l=1,2;j=2,3,l=0,1,2)]} =$

$$\begin{bmatrix} 1.000 & 0.257 & -0.089 & -0.085 & -0.078 & 0.204 & 0.922 & 0.213 \\ 0.257 & 1.000 & -0.091 & -0.086 & -0.081 & 0.228 & 0.204 & 0.922 \\ -0.089 & -0.091 & 1.000 & 0.258 & 0.112 & -0.113 & -0.114 & -0.116 \\ -0.085 & -0.086 & 0.258 & 1.000 & 0.258 & -0.107 & -0.110 & -0.111 \\ -0.078 & -0.081 & 0.112 & 0.258 & 1.000 & -0.104 & -0.104 & -0.107 \\ 0.204 & 0.228 & -0.113 & -0.107 & -0.104 & 1.000 & 0.175 & 0.192 \\ 0.922 & 0.204 & -0.114 & -0.110 & -0.104 & 0.175 & 1.000 & 0.174 \\ 0.213 & 0.922 & -0.116 & -0.111 & -0.107 & 0.192 & 0.174 & 1.000 \end{bmatrix} = 0.0154$$

The second row and column of the 9x9 autocorrelation matrix are $\rho_{it+k,jt+l[(i=1;k=1),(j=1,2,3;l=0,1,2)])}$ and $\rho_{it+k,jt+l[(i=1,2,3;k=0,1,2),(j=1;l=1)]}$. Since (22) is symmetric, the crossed elements of the first row and column can be written as $\rho_{it+k,jt+l[(i=1;k=0),(j=1,2,3;l=0,1,2)]}$ and $\rho^{I}_{it+k,jt+l[(i=1;k=0),(j=1,2,3;l=0,1,2))}$ with the respective minor

$\rho_{1t+k,jt+l[(i=1;k=0,2;i=2,3,k=0,1,2),(j=1,l=0,2;j=2,3,l=0,1,2)]} =$

$$\begin{bmatrix} 1.000 & 0.250 & -0.088 & -0.082 & -0.078 & 0.922 & 0.213 & 0.213 \\ 0.250 & 1.000 & -0.091 & -0.086 & -0.081 & 0.228 & 0.204 & 0.922 \\ -0.088 & -0.091 & 1.000 & 0.258 & 0.112 & -0.113 & -0.114 & -0.116 \\ -0.082 & -0.086 & 0.258 & 1.000 & 0.258 & -0.107 & -0.110 & -0.111 \\ -0.078 & -0.081 & 0.112 & 0.258 & 1.000 & -0.104 & -0.104 & -0.107 \\ 0.922 & 0.228 & -0.113 & -0.107 & -0.104 & 1.000 & 0.175 & 0.192 \\ 0.213 & 0.204 & -0.114 & -0.110 & -0.104 & 0.175 & 1.000 & 0.174 \\ 0.213 & 0.922 & -0.116 & -0.111 & -0.107 & 0.192 & 0.174 & 1.000 \end{bmatrix} = 0.01698$$

The third row and column of the 9x9 autocorrelation matrix are $\rho_{it+k,jt+l[(i=1;k=2),(j=1,2,3;l=0,1,2)]}$ and $\rho_{it+k,jt+l[(i=1,2,3;k=0,1,2),(j=1;l=2)])}$. Since (21) is symmetric, the crossed elements of the first row and column can be written as $\rho_{it+k,jt+l[(i=1;k=2),(j=1,2,3;l=0,1,2)]}$ and $\rho^{I}_{it+k,jt+l[(i=1;k=2),(j=1,2,3;l=0,1,2))}$ with its respective minor

$\rho_{it+k,jt+l[(i=1;k=0,1;i=2,3,k=0,1,2),(j=1,l=0,1;j=2,3,l=0,1,2)])} =$

$$\begin{bmatrix} 1.000 & 0.258 & -0.088 & -0.082 & -0.078 & 0.922 & 0.213 & 0.213 \\ 0.258 & 1.000 & -0.089 & -0.085 & -0.078 & 0.204 & 0.922 & 0.213 \\ -0.088 & -0.089 & 1.000 & 0.258 & 0.112 & -0.113 & -0.114 & -0.116 \\ -0.082 & -0.085 & 0.258 & 1.000 & 0.258 & -0.107 & -0.110 & -0.111 \\ -0.078 & -0.078 & 0.112 & 0.258 & 1.000 & -0.104 & -0.104 & -0.107 \\ 0.922 & 0.204 & -0.113 & -0.107 & -0.104 & 1.000 & 0.175 & 0.192 \\ 0.213 & 0.922 & -0.114 & -0.110 & -0.104 & 0.175 & 1.000 & 0.174 \\ 0.213 & 0.213 & -0.116 & -0.111 & -0.107 & 0.192 & 0.174 & 1.000 \end{bmatrix} = 0.01654$$

The fourth row and column of the 9x9 autocorrelation matrix are $\rho_{it+k,jt+l[(i=2;k=0),(j=1,2,3;l=0,1,2)]}$ and $\rho_{it+k,jt+l[(i=1,2,3;k=0,1,2),(j=2;l=0)]}$. This can be written as $\rho_{it+k,jt+l[(i=2;k=0),(j=1,2,3;l=0,1,2)]}$ and $\rho_{it+k,jt+l[(i=2;k=0),(j=1,2,3;l=0,1,2)]}$ with its respective minor

$\rho_{it+k,jt+l[(i=2;k=1,2;i=1,3,k=0,1,2),(j=2,l=1,2;j=1,3,l=0,1,2)]} =$

$$\begin{bmatrix} 1.000 & 0.258 & 0.250 & -0.082 & -0.078 & 0.922 & 0.213 & 0.213 \\ 0.258 & 1.000 & 0.257 & -0.085 & -0.078 & 0.204 & 0.922 & 0.213 \\ 0.250 & 0.257 & 1.000 & -0.086 & -0.081 & 0.228 & 0.204 & 0.922 \\ -0.082 & -0.085 & 0.086 & 1.000 & 0.258 & -0.107 & -0.110 & -0.111 \\ -0.078 & -0.078 & -0.081 & 0.258 & 1.000 & -0.104 & -0.104 & -0.107 \\ 0.922 & 0.204 & 0.228 & -0.107 & -0.104 & 1.000 & 0.175 & 0.192 \\ 0.213 & 0.922 & 0.204 & -0.110 & -0.104 & 0.175 & 1.000 & 0.174 \\ 0.213 & 0.213 & 0.922 & -0.111 & -0.107 & 0.192 & 0.174 & 1.000 \end{bmatrix} = 0.0026$$

The fifth row and column of the 9x9 autocorrelation matrix are $\rho_{it+k,jt+l[(i=2;k=1),(j=1,2,3;l=0,1,2)]}$ and $\rho_{it+k,jt+l[(i=1,2,3;k=0,1,2),(j=2;l=1)]}$. This can be written as $\rho_{it+k,jt+l[(i=2;k=1),(j=1,2,3;l=0,1,2)]}$ and $\rho^{I}_{it+k,jt+l[(i=2;k=1),(j=1,2,3;l=0,1,2))}$ with its respective minor

$\rho_{it+k,jt+l[(i=2;k=0,2;i=1,3,k=0,1,2),(j=2,l=0,2;j=1,3,l=0,1,2)])} =$

$$\begin{bmatrix} 1.000 & 0.258 & 0.250 & -0.088 & -0.078 & 0.922 & 0.213 & 0.213 \\ 0.258 & 1.000 & 0.257 & -0.089 & -0.078 & 0.204 & 0.922 & 0.213 \\ 0.250 & 0.257 & 1.000 & -0.091 & -0.081 & 0.228 & 0.204 & 0.922 \\ -0.088 & -0.089 & -0.091 & 1.000 & 0.999 & -0.113 & -0.114 & -0.116 \\ -0.078 & -0.078 & -0.081 & 0.999 & 1.000 & -0.104 & -0.104 & -0.107 \\ 0.922 & 0.204 & 0.228 & -0.113 & -0.104 & 1.000 & 0.175 & 0.192 \\ 0.213 & 0.922 & 0.204 & -0.114 & -0.104 & 0.175 & 1.000 & 0.174 \\ 0.213 & 0.213 & 0.922 & -0.116 & -0.107 & 0.192 & 0.174 & 1.000 \end{bmatrix} = 0.0000$$

The minor of the sixth row and column of the matrix is

$$\rho_{it+k,jt+l[(i=2;k=0,1;i=1,3,k=0,1,2),(j=2,l=0,1;j=1,3,l=0,1,2)]} = -0.000000156.$$

The minor of the seventh row and column of the matrix is

$$\rho_{it+k,jt+l[(i=3;k=1,2;i=1,2,k=0,1,2),(j=3,l=1,2;j=1,2,l=0,1,2)]} = -0.000000016.$$

The minor of the eighth row and column of the matrix is

$$\rho_{it+k,jt+l[(i=3;k=0,2;i=1,2,k=0,1,2),(j=3,l=0,2;j=1,2,l=0,1,2)]} = -0.000000016.$$

The minor of the ninth row and column of the matrix is

$$\rho_{it+k,jt+l[(i=3;k=0,1;i=1,2,k=0,1,2),(j=3,l=0,1;j=1,2,l=0,1,2)]} = -0.000000016.$$

$$|\rho_{it+k,jt+l}| = \begin{vmatrix} 1.000 & 0.258 & 0.250 & 0.603 & 0.269 & 0.280 & 0.922 & 0.213 & 0.213 \\ 0.258 & 1.000 & 0.257 & 0.202 & 0.602 & 0.269 & 0.204 & 0.922 & 0.213 \\ 0.259 & 0.257 & 1.000 & 0.153 & 0.202 & 0.602 & 0.228 & 0.204 & 0.922 \\ 0.603 & 0.202 & 0.153 & 1.000 & 0.258 & 0.112 & 0.419 & 0.163 & 0.143 \\ 0.269 & 0.602 & 0.202 & 0.258 & 1.000 & 0.258 & 0.227 & 0.419 & 0.163 \\ 0.280 & 0.269 & 0.602 & 0.112 & 0.258 & 1.000 & 0.259 & 0.226 & 0.419 \\ 0.922 & 0.204 & 0.228 & 0.419 & 0.227 & 0.259 & 1.000 & 0.175 & 0.192 \\ 0.213 & 0.922 & 0.204 & 0.163 & 0.419 & 0.226 & 0.175 & 1.000 & 0.174 \\ 0.213 & 0.213 & 0.922 & 0.413 & 0.163 & 0.419 & 0.192 & 0.174 & 1.000 \end{vmatrix} = 0.00219223$$

The computations of the principal minors show positive and negative cross-autocorrelations. The determinant of the 9x9 autocorrelation matrix with mixed stable and unstable processes have positive values (greater than zero). Not all the principal minors are greater than zero due to mixed processes in the cross-autocorrelations. This explains the fact that the cross-autocovariances/cross-autocorrelations are not from a multivariate stationary process.

4. Conclusion

In a univariate time series, positive definiteness is one of the conditions to ascertain stationarity of a time series process. Stationarity of multivariate time series process is also investigated with similar conditions. One of the conditions for stationarity of a time series process using positive definiteness property is that the principal minors and determinant of the autocorrelations have values greater than zero. In this paper, the cross-autocorrelations of three vector time series were obtained, with two vector series under stable condition and one under unstable condition. The cross-autocorrelations matrix has nine (9) sub-correlation matrices whose stationarity have been investigated using positive definiteness property. From the results, some principal minors of the 9x9 cross-autocorrelation matrix are positive, while some are negative, which show that the cross-correlations are not from stationary multivariate time process. This is due to the fact that urban price index was conditionally unstable, therefore, subjecting the whole multivariate process to non-stationary state. This paper recommends that the stationarity of the sub-autocorrelation matrices which make up the cross-autocorrelation matrix should be primarily investigated. If some of the sub-autocorrelation matrices violates any stationarity condition, it would affect the stability of the whole multivariate time series process as revealed in this paper.

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