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Exponentiated Power Half Logistic Distribution: Theory and Applications

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Abstract. The frequent occurrence of data that may not be adequately modeled using any of the existing distributions has led to the continual introduction of new distributions. One way of introducing a new distribution is to generalize an existing distribution. Among the methods of generalizing distributions, the power transformation and exponential techniques have gained wide acceptability because of their tendency to generate highly flexible distributions. A new distribution called the exponentiated power half logistic distribution is presented in this paper. We establish the relationships between the distribution and some well-known distributions. Raw moment, moment generating function, entropy and order statistics corresponding to the new distribution are discussed. Real life applications of the distribution are also considered. The goodness of fits of the new distribution to three real data sets is compared with that of each of power half logistic distribution, exponentiated half logistic distribution and half logistic distribution using some goodness of fit statistics. The results obtained show that the new model fits each of the data better than the other models.

Keywords: Exponentiation method, flexible distributions, quantile function, reliability function, order statistics.

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1. Introduction

Generalizing distributions is one of the important functions frequently carried out by distribution theorists. It helps one to introduce more flexible distributions than the corresponding parent distributions. The introduced distributions

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often have some desirable properties which the baseline distributions do not have. For example, the generalized Lindley distribution has a bathtub shaped hazard rate unlike the gamma and Weibull distributions and other baseline distributions (Nadarajah et al., 2011). The half logistic distribution discussed in Balakrashinan (1985) is one of the distributions used to model lifetime data. By stating and proving some theorems, Olapade (2003) characterised this distribution. Just like other baseline distributions, it has been generalized using distribution generalization techniques so as to introduce more suitable models for modelling complex data. Olivera et al. (2016) defined and studied the properties of MacDonald half logistic distribution. For the distribution, they derived mean deviations, Renyi entropy, quantile function and the expression for each of the ordinary and incomplete moments. Their numerical results showed that the distribution has the capacity to give better fits to some positive data than the generalized gamma and exponentiated Weibull distributions.

The definition, mathematical properties and application of type 1 half logistic family of distributions were given in Cordeiro et al. (2016). Special cases of the family, namely, type 1 half logistic normal, type 1 half logistic Frechet and type 1 half logistic gamma distributions were presented in the paper. Other cases of this family of distributions are the half logistic Lomax and half logistic generalized Weibull of Anwar and Zahoor (2018) and Anwar and Bibi (2018) respectively. Type II half logistic family of distributions together with their special cases have been presented in Soliman et al (2017).

Seo and Kang (2015) determined moment and maximum likelihood estimators of the parameters of the exponentiated half logistic distribution. Krisharani (2016) considered the power transformation of an exponentiated half logistic distribution and was able to introduce a new distribution. He illustrated the application of the distribution using some real life data. Authors have recently indicated their interest in the generation of new distributions by combining power transformation and exponentiation techniques. Distributions introduced via a combination of the two techniques include the exponentiated power Lindley distribution (Ashour and Eltehiwy, 2015), exponentiated power quasi Lindley (Gica et al, 2017) and exponentiated power Lindley Poisson distribution (Pararai et al, 2017), Exponential Half Logistic-Power Generalized Weibul-G Family of Distributions (Oluyede et al, 2021), Exponentiated Power Function Distribution: Properties and Applications (Arshad et al, 2020), Exponentiated quasi power Lindley power series distribution with applications in medical science (Anwar et al, 2020), the exponentiated power generalized Weibull: Properties and Application (Pena-Ramirez et al, 2018). These distributions were proven to be more flexible than their sub models. In this paper, the exponentiated power half logistic distribution (EPHLD) is defined. Its properties and applications are also given. An interesting rationale for considering this distribution is that if the power half logistic random variables $X_1, X_2, ..., X_n$ are failure times of the components of a system, assumed to be independent, the probability that system fails before time x is essentially the cdf of the exponentiated power half logistic (EPHL) variable.

2. Materials and Methods

2.1 *The pdf and cdf of EPHL variable*

If X follows an exponentiated distribution based on a baseline distribution with probability density function (pdf) and cumulative density function (cdf) given as $f(x)$ and $F(x)$ respectively, then $F(x) = (G(x))^{\alpha}$ and $f(x) = \alpha g(x)(G(x))^{\alpha-1}$, where $\alpha > 0$, $g(x)$ and $G(x)$ are respectively the pdf and cdf of the baseline distribution. In the case of the power half logistic distribution, the pdf is

$$g(x) = \frac{2\theta\beta e^{\beta x^{\theta}} x^{\theta-1}}{(e^{\beta x^{\theta}} + 1)^2}, x > 0, \beta, \theta > 0 \tag{1}$$

while the cdf is

$$G(x) = \frac{e^{\beta x^{\theta}} - 1}{e^{\beta x^{\theta}} + 1}, x > 0, \beta, \theta > 0. \tag{2}$$

Consequently, the cdf of the exponentiated power half logistic (EPHL) variable (X) becomes

$$F(x) = \left(\frac{e^{\beta x^{\theta}} - 1}{e^{\beta x^{\theta}} + 1}\right)^{\alpha}, x > 0, \alpha, \beta, \theta > 0. \tag{3}$$

Using (3), the pdf of the EPHL variable is of the form

$$f(x) = \frac{2\alpha\theta\beta e^{\beta x^{\theta}} x^{\theta-1} \left(e^{\beta x^{\theta}} - 1\right)^{\alpha-1}}{\left(e^{\beta x^{\theta}} + 1\right)^{\alpha+1}}, \quad x > 0, \alpha, \beta, \theta > 0 \tag{4}$$

Figure 1 shows the plots of the pdf of the EPHL variable based on several sets of values of the parameters of the distribution.

The special cases of the EPHLD are the power half logistic distribution, exponential half logistic distribution (EHLD) and half logistic distribution (HLD). If α = 1, we have the PHLD. For β = 1, the EPHLD results in the EHLD while the EPHLD amounts to the (HLD) whenever α = 1 and θ = 1. Again, if X has the EPHLD with parameters α, β and θ, we write $X \sim EPHLD\,(\alpha, \beta, \theta)$.

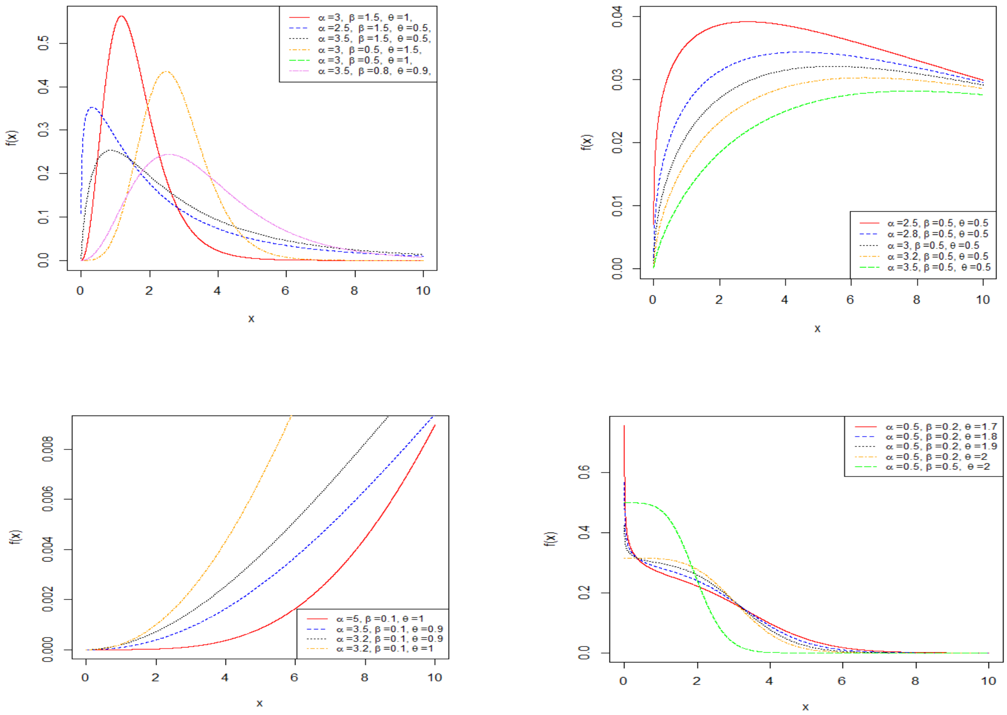

Figure 1: Plots of the pdf of the EPHL variable

2.2 *Relationships with other distributions*

Here, we characterise the EPHLD through theorems that relates it to other univariate distributions. Suppose X is an exponentiated power half logistic (EPHL) variable, we state and prove the following theorems.

THEOREM 2.1 *If*

$$W = \ln\left(\frac{e^{\beta X^{\theta}}+1}{e^{\beta X^{\theta}}-1}\right)^{\alpha} \tag{5}$$

W is an exponential variable with mean 1.

Proof. Let $w = \ln\left(\frac{e^{\beta x^{\theta}}+1}{e^{\beta x^{\theta}}-1}\right)$. Then $x = \left(\frac{1}{\beta}\ln\left(\frac{1+e^{\frac{w}{\alpha}}}{e^{\frac{w}{\alpha}}-1}\right)\right)^{\frac{1}{\theta}}$. The Jacobian of the transformation is

$$J = \frac{-2e^{\frac{w}{\alpha}}\left[\ln\left(\frac{1+e^{\frac{w}{\alpha}}}{e^{\frac{w}{\alpha}}-1}\right)\right]^{\frac{1}{\theta}-1}}{\beta^{\frac{1}{\theta}}\theta\left(e^{\frac{2w}{\alpha}}-1\right)}.$$

Therefore, the probability density function (pdf) of W is

$$h\left(w\right)=e^{-w},w>0.$$

which is the pdf of an exponentially distributed random variable with mean 1. ■

THEOREM 2.2 *If*

$$W=\delta\left(\frac{e^{\beta X^{\theta}}+1}{e^{\beta X^{\theta}}-1}\right), \tag{6}$$

W has type-I Pareto distribution with parameters α and δ.

Proof. Let $\omega=\delta\left(\frac{e^{\beta X^{\theta}}+1}{e^{\beta X^{\theta}}-1}\right)$. Then $x=\left(\frac{1}{\beta}\ln\left(\frac{\omega+\delta}{\omega-\delta}\right)\right)^{\frac{1}{\theta}}$. The Jacobian of the transformation is

$$J=\frac{-2\delta\left(\ln\left(\frac{\omega+\delta}{\omega-\delta}\right)\right)^{\frac{1}{\theta}-1}}{\beta^{\frac{1}{\theta}}\theta\left(\omega^{2}-\delta^{2}\right)}.$$

Consequently, the pdf of W is

$$h(\omega)=\alpha\delta^{\alpha}\omega^{-\alpha-1},\alpha>0,\delta>0,\omega>\delta$$

which is an indication that W follows a type-I Pareto distribution with parameters α and δ. ■

THEOREM 2.3 *If*

$$W=\delta\left(\alpha\ln\left(\frac{e^{\beta X^{\theta}}+1}{e^{\beta X^{\theta}}-1}\right)\right)^{\frac{1}{\lambda}}, \tag{7}$$

W has a two-parameter Weibull distribution with parameters λ and δ.

Proof. Let

$$\omega=\delta\left(\alpha\ln\left(\frac{e^{\beta X^{\theta}}+1}{e^{\beta X^{\theta}}-1}\right)\right)^{\frac{1}{\lambda}}.$$

Then

$$x=\left[\frac{1}{\beta}\ln\left[\frac{\exp\left(\frac{1}{\alpha}\left(\frac{\omega}{\delta}\right)^{\lambda}\right)+1}{\exp\left(\frac{1}{\alpha}\left(\frac{\omega}{\delta}\right)^{\lambda}\right)-1}\right]\right]^{\frac{1}{\theta}}.$$

Then Jacobian of this transformation is

$$J=\frac{-2\lambda\left(\frac{\omega}{\delta}\right)^{\lambda}\left(\ln\left(\frac{\exp\left(\frac{1}{\alpha}\left(\frac{\omega}{\delta}\right)^{\lambda}\right)+1}{\exp\left(\frac{1}{\alpha}\left(\frac{\omega}{\delta}\right)^{\lambda}\right)-1}\right)\right)^{\frac{1}{\theta}-1}}{\omega\alpha\beta^{\frac{1}{\theta}}\theta\left(\exp\left(\frac{2}{\alpha}\left(\frac{\omega}{\delta}\right)^{\lambda}\right)-1\right)}.$$

Thus, the pdf of W is

$$h(w)=\frac{\lambda}{\delta}\left(\frac{\omega}{\delta}\right)^{\lambda-1}\exp\left(-\left(\frac{\omega}{\delta}\right)^{\lambda}\right),,\delta>0,\lambda>0,\omega>0$$

indicating that W follows the Weibull distribution. ■

THEOREM 2.4 *If*

$$W=\mu-\sigma\ln\left(\left(\frac{e^{\beta X^{\theta}}+1}{e^{\beta X^{\theta}}-1}\right)^{\alpha}-1\right), \tag{8}$$

W follows a logistic distribution with parameters μ and σ.

Proof. Let

$$w=\mu-\sigma\ln\left(\left(\frac{e^{\beta x^{\theta}}+1}{e^{\beta x^{\theta}}-1}\right)^{\alpha}-1\right).$$

Then

$$x=\left(\frac{1}{\beta}\ln\left(\frac{(\exp(-(\frac{w-\mu}{\sigma}))+1)^{\frac{1}{\alpha}}+1}{(\exp(-(\frac{w-\mu}{\sigma}))+1)^{\frac{1}{\alpha}}-1}\right)\right)^{\frac{1}{\theta}}.$$

The Jacobian of this transformation is

$$J=\frac{2\exp\left(-\left(\frac{w-\mu}{\sigma}\right)\right)\left(\exp\left(-\left(\frac{w-\mu}{\sigma}\right)+1\right)\right)^{\frac{1}{\alpha}-1}\left(\ln\left(\frac{\left(\exp\left(-\left(\frac{w-\mu}{\sigma}\right)\right)+1\right)^{\frac{1}{\alpha}}+1}{\left(\exp\left(-\left(\frac{w-\mu}{\sigma}\right)\right)+1\right)^{\frac{1}{\alpha}}-1}\right)\right)^{\frac{1}{\theta}-1}}{\alpha\beta^{\frac{1}{\theta}}\sigma\theta\left(\left(\exp\left(-\left(\frac{w-\mu}{\sigma}\right)\right)+1\right)^{\frac{2}{\alpha}}-1\right)}.$$

Hence, the pdf of W is

$$h\left(w\right)=\frac{\exp\left(-\left(\frac{w-\mu}{\sigma}\right)\right)}{\sigma\left(1+\exp\left(-\left(\frac{w-\mu}{\sigma}\right)\right)\right)^{2}},-\infty<w<\infty,-\infty<\mu<\infty,\sigma>0,$$

which is the logistic pdf. ■

THEOREM 2.5 *If*

$$W=\mu-\sigma\ln\left(\alpha\ln\left(\frac{e^{\beta X^{\theta}}+1}{e^{\beta X^{\theta}}-1}\right)\right), \tag{9}$$

W follows a Gumbel distribution with parameters μ and σ.

Proof. Let

$$w=\mu-\sigma\ln\left(\alpha\ln\left(\frac{e^{\beta X^{\theta}}+1}{e^{\beta X^{\theta}}-1}\right)\right).$$

Then

$$x=\left(\frac{1}{\beta}\ln\left(\frac{\exp(\frac{1}{\alpha}\exp(-(\frac{w-\mu}{\sigma})))+1}{\exp(\frac{1}{\alpha}\exp(-(\frac{w-\mu}{\sigma})))-1}\right)\right)^{\frac{1}{\theta}}.$$

The Jacobian of this transformation is

$$J=\frac{2\exp(\frac{1}{\alpha}\exp(-(\frac{w-\mu}{\sigma})))\exp(-(\frac{w-\mu}{\sigma}))\left(\ln\left(\frac{\exp(\frac{1}{\alpha}\exp(-(\frac{w-\mu}{\sigma})))+1}{\exp(\frac{1}{\alpha}\exp(-(\frac{w-\mu}{\sigma})))-1}\right)\right)^{\frac{1}{\theta}-1}}{\alpha\beta^{\frac{1}{\theta}}\sigma\theta\left(\exp(\frac{2}{\alpha}\exp(-(\frac{w-\mu}{\sigma})))-1\right)}.$$

Hence, the pdf of W is

$$h\left(w\right)=\frac{1}{\sigma}\exp\left(-\left(\frac{w-\mu}{\sigma}\right)-\exp\left(-\left(\frac{w-\mu}{\sigma}\right)\right)\right),$$

$-\infty<w<\infty,-\infty<\mu<\infty,\sigma>0$, which is the pdf associated with a Gumbel distribution. ■

2.3 *Raw moments, moment generating function and Renyi entropy of an EPHL variable*

Moments of a random variable provide vital information such as mean, variance and coefficient of skewness and kurtosis of a random variable. Let $X=EPHLD(\alpha,\beta,\theta)$. The rth raw moment of X is

$$
\begin{aligned}
E(X^r) &= \int_0^\infty x^r f(x)\, dx \\
&= 2\alpha\beta\theta \int_0^\infty \frac{x^{r+\theta-1} e^{\beta x^\theta} \left(e^{\beta x^\theta} - 1\right)^{\alpha-1}}{\left(e^{\beta x^\theta} + 1\right)^{\alpha+1}} dx \\
&= 2\alpha\beta\theta \int_0^\infty \frac{x^{r+\theta-1} e^{-\beta x^\theta} \left(1 - e^{-\beta x^\theta}\right)^{\alpha-1}}{\left(1 + e^{-\beta x^\theta}\right)^{\alpha+1}} dx.
\end{aligned} \tag{10}
$$

To evaluate the integral in (10), we use the following binomial series expansions:

$$
\left(1 + e^{-\beta x^\theta}\right)^{-(\alpha+1)} = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i}{i} e^{-\beta i x^\theta}; \tag{11}
$$

$$
\left(1 - e^{-\beta x^\theta}\right)^{\alpha-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha+1}{j} e^{-\beta j x^\theta}. \tag{12}
$$

Thus,

$$
\begin{aligned}
E\left(X^r\right) &= 2\alpha\beta\theta \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} (-1)^{i+j} \binom{\alpha+i}{i} \binom{\alpha-1}{j} \int_0^\infty x^{r+\theta-1} e^{-\beta(i+j+1)x^\theta} dx \\
&= 2\alpha \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} (-1)^{i+j} \binom{\alpha+i}{i} \binom{\alpha-1}{j} \frac{\Gamma\left(\frac{r+\theta}{\theta}\right)}{\beta^{\frac{1}{\theta}} (i+j+1)^{\frac{r+\theta}{\theta}}}
\end{aligned} \tag{13}
$$

The moment generating function of an EPHL variable has the form

$$
\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \sum_{k=0}^{\infty} \frac{t^k E(X^k)}{K!} \\
&= 2\alpha \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} (-1)^{i+j} \binom{\alpha+i}{i} \binom{\alpha-1}{j} \frac{t^k \Gamma\left(\frac{k+\theta}{\theta}\right)}{\beta^{\frac{k}{\theta}} (i+j+1)^{\frac{k+\theta}{\theta}} \Gamma(k+1)}
\end{aligned} \tag{14}
$$

Several measures of uncertainty in a random have been introduced. The Renyi entropy is certainly popular and widely used among these measures. For an EPHL variable X, Renyi entropy is defined by

$$
\begin{aligned}
I_R(\delta) &= \frac{1}{\delta-1}\log\int_0^\infty f^\delta(x)dx, \delta>0 \text{ and } \delta\neq 1 \\
&= \frac{1}{\delta-1}(\delta\log 2+\delta\log\alpha+\delta\log\beta+\delta\log\theta)+\frac{1}{\delta-1}\times \\
&\log\left(\int_0^\infty \frac{x^{(\theta-1)\delta}e^{-\beta\delta x^\theta}\left(1-e^{-\beta x^\theta}\right)^{(\alpha-1)\delta}}{\left(1+e^{-\beta x^\theta}\right)^{(\alpha+1)\delta}}dx\right) \\
&= \frac{1}{\delta-1}(\delta\log 2+\delta\log\alpha+\delta\log\beta+\delta\log\theta) \\
&+\frac{1}{\delta-1}\log\left(\sum_{j=0}^\infty\sum_{i=0}^\infty(-1)^{i+j}\binom{(\alpha+1)\delta+i-1}{i}\binom{(\alpha-1)\delta}{j}\times\right. \\
&\left.\int_0^\infty x^{(\theta-1)\delta}e^{-\beta(i+j+\delta)x^0}dx\right) \\
&= \frac{1}{\delta-1}(\log 2+\log\alpha+\log\beta+\log\theta) \\
&+\frac{1}{\delta-1}\log\left(\sum_{j=0}^\infty\sum_{i=0}^\infty(-1)^{i+j}\binom{(\alpha+1)\delta+i-1}{i}\binom{(\alpha-1)\delta}{j}\times\right. \\
&\left.\frac{\Gamma\left(\frac{(\theta-1)\delta+1}{\theta}\right)}{\theta(\beta(i+j+\delta))^{\frac{(\theta-1)\delta+1}{\theta}}}\right)
\end{aligned}
$$

2.4 *Reliability analysis*

Two useful functions in reliability analysis are the survival function $(R(x))$ and the hazard rate function $(h(x))$. With $f(x)$ and $F(x)$ representing the pdf and cdf, respectively, of a EPHL variable, then

$$
\begin{aligned}
R(x) &= 1-F(x) \\
1 &- \left(\frac{e^{\beta x^\theta}-1}{e^{\beta x^\theta}+1}\right)^\alpha
\end{aligned}
$$

On the other hand, the hazard rate function is
$h(x) = \frac{f(x)}{1-F(x)}$

$$= \frac{2\alpha\theta\beta e^{\beta x^{\theta}}\left(e^{\beta x^{\theta}}-1\right)^{\alpha-1}}{\left(e^{\beta x^{\theta}}+1\right)^{\alpha+1}\left(1-\left(\frac{1-e^{-\beta x^{\theta}}}{1+e^{-\beta x^{\theta}}}\right)^{\alpha}\right)}.$$

2.5 *Quantile and generation of random numbers*

The quantile function x_q of the EPHLD is determined by solving the equation $F(x_q) = q, q \in (0,1)$. Thus we have

$$x_q = \left(\frac{1}{\beta}\ln\left(\frac{1+q^{\frac{1}{\alpha}}}{1-q^{\frac{1}{\alpha}}}\right)\right)^{\frac{1}{\theta}}. \tag{15}$$

By substituting $q = 0.25$, 0.5 and 0.75 into (15), we obtain the first quartile, median and third quartile of the distribution respectively. If U is a uniformly distributed random variable on the interval (0, 1) and u is any observed value of U, an expression that can be used to generate random numbers from EPHL distribution is derived using the inverse cdf technique. Finding this expression amounts to solving for x in the equation:

$$F(x) = u \tag{16}$$

The solution to (16) is

$$x = \left(\frac{1}{\beta}\ln\left(\frac{1+u^{\frac{1}{\alpha}}}{1-u^{\frac{1}{\alpha}}}\right)\right)^{\frac{1}{\theta}} \tag{17}$$

2.6 *Estimation of parameters*

When we have a random sample from an EPHL distribution with parameters α, β and θ, the likelihood function is

$$\begin{aligned} L(\alpha,\beta,\theta) &= \prod_{i=1}^{n} f(x_i) \\ &= (2\alpha\beta\theta)^n \prod_{i=1}^{n}\left(\frac{x_i^{\theta-1}e^{\beta x_i^{\theta}}\left(e^{\beta x^{\theta}}-1\right)^{\alpha-1}}{\left(e^{\beta x_i^{\theta}}+1\right)^{\alpha+1}}\right) \end{aligned} \tag{18}$$

With (18), we obtain the log-likelihood function

$\ln L(\alpha,\beta,\theta) = n\ln 2 + n\ln\alpha + n\ln\beta + n\ln\theta + (\theta-1)\sum_{i=1}^{n}\ln x_i + \beta\sum_{1=1}^{n} x_i^{\theta} + (\alpha-1)\sum_{i=1}^{n}\ln\left(e^{\beta x_i^{\theta}}-1\right) - (\alpha+1)\sum_{i=1}^{n}\ln\left(e^{\beta x_i^{\theta}}+1\right).$

Partial derivatives of $\ln L(\alpha,\beta,\theta)$ with respect to the parameters are as follows:

$\frac{\partial \ln L(\alpha,\beta,\theta)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(e^{\beta x_i^{\theta}}-1\right) - \sum_{i=1}^{n}\ln\left(e^{\beta x_i^{\theta}}+1\right);$

$\frac{\partial \ln L(\alpha,\beta,\theta)}{\partial\beta} = \frac{n}{\beta} + \sum_{1=1}^{n} x_i^{\theta} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\theta}e^{\beta x_i^{\theta}}}{e^{\beta x_i^{\theta}}-1} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i^{\theta}e^{\beta x_i^{\theta}}}{e^{\beta x_i^{\theta}}+1};$

$\frac{\partial \ln L(\alpha,\beta,\theta)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln x_i + \beta\sum_{1=1}^{n} x_i^{\theta}\ln x_i + \beta(\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\theta}\ln x_i e^{\beta x_i^{\theta}}}{e^{\beta x_i^{\theta}}-1} - \beta(\alpha+1)\sum_{i=1}^{n}\frac{x_i^{\theta}\ln x_i e^{\beta x_i^{\theta}}}{e^{\beta x_i^{\theta}}+1}.$

Parameters of the EPHL distribution can be estimated by solving simultaneously the nonlinear equations:

$$\frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(e^{\beta x_i^{\theta}}-1\right) - \sum_{i=1}^{n}\ln\left(e^{\beta x_i^{\theta}}+1\right) = 0; \tag{19}$$

$$\frac{n}{\beta} + \sum_{i=1}^{n} x_i^{\theta} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\theta}e^{\beta x_i^{\theta}}}{e^{\beta x_i^{\theta}}-1} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i^{\theta}e^{\beta x_i^{\theta}}}{e^{\beta x_i^{\theta}}+1} = 0; \tag{20}$$

$$\frac{n}{\theta} + \sum_{i=1}^{n}\ln x_i + \beta\sum_{i=1}^{n} x_i^{\theta}\ln x_i + \beta(\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\theta}\ln x_i e^{\beta x_i^{\theta}}}{e^{\beta x_i^{\theta}}-1}$$

$$-\beta(\alpha+1)\sum_{i=1}^{n}\frac{x_i^{\theta}\ln x_i e^{\beta x_i^{\theta}}}{e^{\beta x_i^{\theta}}+1} = 0. \tag{21}$$

Suffice it to say that only numerical solution of (19) to (21) can be found. R packages may be used to find the requisite estimates of the parameters based on an appropriate data set. Sometimes, we need to find a confidence interval for any of the parameters. To this effect, the standard error estimate of the parameter should be known. The asymptotic distribution of the maximum likelihood estimator vector $\hat{\Theta} = \left(\hat{\alpha},\hat{\beta},\hat{\theta}\right)^T$ is also required. Let $\Theta = (\alpha,\beta,\theta)^T$. Then, the asympitic distribution of $\left(\hat{\Theta}-\Theta\right)$ is a multivariate normal distribution with mean vector 0 and variance-covariance matrix $I(\Theta)^{-1}$. Here $I(\Theta)^{-1}$ is the inverse of 3×3 expected information matrix. In practice, $I(\Theta)$ is approximated by $J(\hat{\Theta})$, which is a 3×3 observed information matrix evaluated at $\Theta=\hat{\Theta}$.

Given that

$$J\left(\hat{\Theta}\right)^{-1}=\begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33}\end{pmatrix}=\begin{pmatrix}\hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13}\\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23}\\ \hat{A}_{31} & \hat{A}_{32} & \hat{A}_{33}\end{pmatrix}^{-1},$$

we have

$$\hat{A}_{11}=\frac{-\partial^2\ln L(\alpha,\beta,\theta)}{\partial\alpha^2}\Big|_{\Theta=\hat{\Theta}},\quad \hat{A}_{22}=\frac{-\partial^2\ln L(\alpha,\beta,\theta)}{\partial\beta^2}\Big|_{\Theta=\hat{\Theta}},$$

$$\hat{A}_{33}=\frac{-\partial^2\ln L(\alpha,\beta,\theta)}{\partial\theta^2}\Big|_{\Theta=\hat{\Theta}},\quad \hat{A}_{12}=\frac{-\partial^2\ln L(\alpha,\beta,\theta)}{\partial\alpha\partial\beta}\Big|_{\Theta=\hat{\Theta}},$$

$$\hat{A}_{13}=\frac{-\partial^2\ln L(\alpha,\beta,\theta)}{\partial\alpha\partial\theta}\Big|_{\Theta=\hat{\Theta}}\text{ and }\hat{A}_{23}=\frac{-\partial^2\ln L(\alpha,\beta,\theta)}{\partial\beta\partial\theta}\Big|_{\Theta=\hat{\Theta}}$$

The approximate $100(1-\eta)$ confidence intervals for α, β and θ are $\hat{\alpha}\pm z_{\frac{\eta}{2}}\sqrt{\hat{V}_{11}}$, $\hat{\beta}\pm z_{\frac{\eta}{2}}\sqrt{\hat{V}_{22}}$ and $\hat{\theta}\pm z_{\frac{\eta}{2}}\sqrt{\hat{V}_{33}}$ respectively. It is noteworthy that $z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}th$ percentile of the standard normal distribution.

2.7 *Order statistics*

Many problems arising from statistical theory and applications are solved using the concept of order statistics. Let $X_1, X_2, .., X_n$ be a random sample from EPHL distribution. The pdf of the jth order statistic $(X_{j:n})$ is

$f_{X_{j:n}}(x)=\frac{n!}{(j-1)!(n-j)!}f(x)(F(x))^{j-1}(1-F(x))^{n-j}$

$$=\frac{2\alpha\beta\theta n!x^{\theta-1}e^{\beta x^\theta}\left(e^{\beta x^\theta}-1\right)^{\alpha-1}}{(j-1)!(n-j)!\left(e^{\beta x^\theta}+1\right)^{\alpha+1}}\left(\left(\frac{e^{\beta x^\theta}-1}{e^{\beta x^\theta}+1}\right)^{\alpha}\right)^{j-1}\left(1-\left(\frac{e^{\beta x^\theta}-1}{e^{\beta x^\theta}+1}\right)^{\alpha}\right)^{n-j} \tag{22}$$

Substituting $j = 1$ into (22), the pdf of the first order statistic is found to be

$$f_{X_{1:n}}(x) = \frac{2\alpha\beta\theta n x^{\theta-1} e^{\beta x^\theta} \left(e^{\beta x^\theta} - 1\right)^{\alpha-1}}{\left(e^{\beta x^\theta} + 1\right)^{\alpha+1}} \left(1 - \left(\frac{e^{\beta x^\theta} - 1}{e^{\beta x^\theta} + 1}\right)^{\alpha}\right)^{n-1}. \tag{23}$$

Similarly, with $j = n$, we find the pdf of the nth order statistic to be

$$f_{X_{n:n}}(x) = \frac{2\alpha\beta\theta n x^{\theta-1} e^{\beta x^\theta} \left(e^{\beta x^\theta} - 1\right)^{\alpha-1}}{\left(e^{\beta x^\theta} + 1\right)^{\alpha+1}} \left(\left(\frac{e^{\beta x^\theta} - 1}{e^{\beta x^\theta} + 1}\right)^{\alpha}\right)^{n-1}. \tag{24}$$

3. Results and Discussion

3.1 *Simulation study*

Some simulations are conducted for the purpose of examining the performance of the MLEs of the EPHL distribution parameters. Based on the sample sizes 50, 100,150, 500 and two different sets of the parameter values, data are simulated from the EPHL distribution using the inverse cdf technique. With the help of R software, we are able to simulate 1000 samples under the stated conditions. The two sets of parameter values are designated by Case I: $\alpha = 1.5,\ \beta = 0.8,\ \theta = 0.2$ and Case II: $\alpha = 1.8,\ \beta = 2.5,\ \theta = 0.7$. The average estimate (AE), average bias (AB) and root mean squared error (RMSE) corresponding to each set of parameter values , sample size (n) and 1000 samples are calculated and presented in Table 1.

Table 1: Simulation results based on 1000 samples from the EPHL distribution under various sample sizes and parameter values.

N	2*Parameter	Case I			Case II		
		AE	AB	RMSE	AE	AB	RMSE
3*50	α	1.8887	0.3887	1.8134	2.6156	0.8156	3.8888
	β	0.8295	0.0295	0.3400	2.5272	0.0272	0.6760
	θ	0.2352	0.03518	0.4057	0.8723	0.1723	4.2775
3*100	α	1.6796	0.1796	0.8926	2.1345	0.3345	1.7301
	β	0.8283	0.0283	0.8953	2.5499	0.0499	1.5795
	θ	0.2123	0.0123	0.3891	0.7371	0.0371	1.1740
3*150	α	1.5905	0.0905	0.6077	1.9633	0.1633	0.8389
	β	0.8099	0.0099	0.3116	2.5262	0.0260	0.8283
	θ	0.2094	0.0094	0.2982	0.7253	0.0253	0.7987
3*500	α	1.5303	0.0303	0.2817	1.8359	0.0359	0.3716
	β	0.8075	0.0075	0.2379	2.5075	0.0075	0.2358
	θ	0.2017	0.0017	0.0545	0.7084	0.0084	0.2665

Expectedly, the results in Table 1 show that the parameter estimates tend towards the actual parameter values as the sample size increases. Also, average bias and root mean squared error tend to zero with increasing sample size.

3.2 *Application*

In this section, we fit the EPHL distribution to three data sets in order to demonstrate its flexibility, practical applications and frequent capability to yield better fit to certain data than any of its special cases. The first dataset comprises of the number of successive failures of the conditioning of each member in a fleet of 13 Boeing 720 jet airplanes as reported in Proschan (1963) and subsequently modeled by Huang and Olueye (2014). It is given below:

194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71.

The second data set constitutes the survival times of patients suffering from head and neck cancer, which was reported by Effron (1988) and subsequently analysed by Shanker et al. (2016). It is presented below:

6.53, 7, 10.42, 14.48, 16.10, 22.70, 34, 41.55, 42, 45.28, 49.40, 53.62, 63, 64, 83, 84, 91, 108,112, 129, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273, 277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417.

The third data set comprising levels of GAG in urine of children is one of R in-built data. The pdfs corresponding to PHLD and EPHLD are stated in (2.1) and (2.4) respectively. Let the pdfs associated with EHLD and HLD be $f_1(x)$ and $f_2(x)$ respectively. Following Usman et al. (2017), we have

$$f_1(x) = \frac{2\theta e^{-\theta x}}{(1+e^{-\theta x})^2}, x > 0, \theta > 0$$

and

$$f_2(x) = \frac{2\alpha\theta e^{-\theta x}\left(1-e^{-\theta x}\right)^{\alpha-1}}{(1+e^{-\theta x})^{\alpha+1}}, x > 0, \alpha, \theta > 0.$$

The comparison of the distributions is based on Akaike information criteria (AIC), Bayesian information criteria (BIC) and goodness of fit measures like the Kolmogorov-Smirnov Statistic (KS), Anderson-Darling statistic (A*) and Cramer-Von Mises Statistic (W*).

Notably,

$$AIC = -2\hat{l} + 2k$$

$$BIC = -2\hat{l} + k\ln(n)$$

and

$$KS = \sup_x |F(x) - F_n(x)|,$$

where $\hat{l}$ is the log-likelihood function evaluated at maximum likelihood estimates of the parameters of the distribution under consideration, k is the number of parameters to be estimated, n is the sample size and $F_n(x)$ is the empirical distribution function. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be an ordered sample from the $EPHLD(\alpha, \beta, \theta)$. Following Dey et al. (2017), we have

$$A^* = -n - \tfrac{1}{n}\sum_{i=1}^{n}(2i-1)(\log(F(x_{(i)}, \hat{\alpha}, \hat{\beta}, \hat{\theta}) + \log(1-F(x_{(i)}, \hat{\alpha}, \hat{\beta}, \hat{\theta})))^2$$

and

$$W^* = \tfrac{1}{12n} + \sum_{i=1}^{n}\left(\log(F(x_{(i)}, \hat{\alpha}, \hat{\beta}, \hat{\theta}) - \tfrac{2i-1}{2n}\right)^2$$

As usual, the distribution with the best fit among all the distributions fitted to a particular data set is the distribution that is associated with the minimum AIC, BIC, KS, A^* and W^*. Tables 2, 3 and 4 contain maximum likelihood estimates of the parameters of the four models and their corresponding $AIC, BIC, KS, (A^*)$ and (W^*) values based on each of the three data.

Results presented in Tables 2, 3 and 4 clearly show that the exponentiated power half logistic distribution corresponds to minimum AIC, BIC, KS, A^* and W^* values, making it the best among all the distributions fitted to the three data under consideration.

4. Conclusion

The exponential power half logistic distribution (EPHLD), which has as its special cases the power half logistic, exponentiated half logistic and half logistic distributions has been introduced. Explicit expressions have been given for the pdf, cdf, raw moments, moment generating function, reliability function and hazard function among other properties of the distribution. The distribution was graphically shown to be capable of having distinct and some unique shapes, depending on the values of the parameters. In particular, the pdf of the distribution is L-shaped if $\alpha = 0.1$, $\beta = 0.01$, $\theta = 0.005$. The hazard rate function also has different shapes corresponding to different sets of the parameters. With regards to some real data sets, we established the flexibility and applicability of the distribution. As evident in the numerical results obtained, the EPHLD provided better fits than each of its special cases.

Table 2: Parameter estimates and model selection criteria for distributions fitted to the first data

Distribution	Estimate	$-\hat{l}$	AIC	BIC	KS	A^*	W^*
$EPHLD(\alpha,\beta,\theta)$	$\hat{\alpha}=2.8038$	410.1358	826.2716	834.8277	0.0408	0.1962	0.0291
	$\hat{\beta}=0.7630$						
	$\hat{\theta}=0.5476$						
$PHLD(\beta,\theta)$	$\hat{\beta}=0.0475$	1039.484	2082.968	2089.441	0.0627	1.2923	0.1869
	$\hat{\theta}=0.7679$						
$EHLD(\alpha,\theta)$	$\hat{\alpha}=0.7248$	1044.1560	2092.312	2098.7850	0.1017	2.7812	0.5247
	$\hat{\theta}=0.0120$						
$HLD(\theta)$	$\hat{\theta}=0.0145$	1051.0180	2104.037	2107.273	0.1535	8.5693	1.6161
[0ex]							

Table 3: Parameter estimates and model selection criteria for distributions fitted to the second data

Distribution	Estimate	$-\hat{l}$	AIC	BIC	KS	A^*	W^*
$EPHLD(\alpha,\beta,\theta)$	$\hat{\alpha}=2.7722$	370.5420	747.0840	753.2653	0.1336	0.9405	0.1943
	$\hat{\beta}=0.1640$						
	$\hat{\theta}=0.5121$						
$PHLD(\beta,\theta)$	$\hat{\beta}=0.0175$	372.8764	749.7529	753.8737	0.1573	1.3302	0.2540
	$\hat{\theta}=0.8233$						
$EHLD(\alpha,\theta)$	$\hat{\alpha}=0.8455$	374.4559	752.9117	757.0326	0.1963	1.6828	0.3306
	$\hat{\theta}=0.0055$						
$HLD(\theta)$	$\hat{\theta}=0.0061$	374.9681	751.9362	753.9966	0.2137	2.0352	0.3957
[1ex]							

Table 4: Parameter estimates and model selection criteria for distributions fitted to the third data

Distribution	Estimate	$-\hat{l}$	AIC	BIC	KS	A^*	W^*
$EPHLD(\alpha,\beta,\theta)$	$\hat{\alpha}=6.7234$	1060.7860	2127.5730	2138.8210	0.0376	0.4060	0.0711
	$\hat{\beta}=0.7263$						
	$\hat{\theta}=0.5921$						
$PHLD(\beta,\theta)$	$\hat{\beta}=0.0404$	1080.2570	2164.5130	2172.0120	0.0682	3.1988	0.4546
	$\hat{\theta}=1.3392$						
$EHLD(\alpha,\theta)$	$\hat{\alpha}=2.0157$	1067.5550	2139.1090	2146.6080	0.0738	1.7105	0.3105
	$\hat{\theta}=0.1535$						
$HLD(\theta)$	$\hat{\theta}=0.1097$	1099.3830	2200.7650	2204.5150	0.1625	11.2058	1.7359
[1ex]							

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