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Another look at transmuted Weibull-Rayleigh distribution and its properties

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Abstract. Recently, attention has been shifted by many authors to new and more flexible distributions useful for modelling life time data. Hence, in this paper, a new four- parameter lifetime model, called the Transmuted Weibull Rayleigh distribution (TWRD), an extension of Weibull Rayleigh distribution, is defined and studied. The approach of Quadratic Rank Transmutation Map (QRTM) is used to obtain a new distribution. Some of its properties such as reliability function, moments, quantile function and order statistics are derived and the method of maximum likelihood estimation is used for estimating the model parameters. It was observed that, the probability density takes many forms such as symmetric, asymmetric, unimodal or bathtub depending on the values of the parameters. The usefulness of the transmuted Weibull Rayleigh distribution for modelling data is illustrated using real data.

Keywords: QRTM, TWRD, reliability function, moments, order statistics, maximum likelihood estimation.

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1. Introduction

In this article we introduce a new life time model referred to as transmuted Weibull-Rayleigh distribution, a generalization of Rayleigh distribution. The Rayleigh distribution is a special case of the two parameter Weibull distribution with the shape parameter equal to 2. It was derived by Rayleigh (1980) from the amplitude of sound resulting from many important sources. It has a wide range of applications including life testing experiments, reliability analysis, applied statistics and clinical studies. Despite its applicability, the distribution like several others, suffers problem of lack of flexibility due to having only one parameter, hence, there is need to generalize the distribution to come up with compound distributions that are more flexible.

In recent years, attention has been shifted by many authors to new and more flexible distributions useful for modelling life time data. These new distributions are generated using different method of generating distribution as was reviewed by Lee et al. (2013) and Tahir and Nasarajah (2015). Many of the distributions such as Normal, Exponential, Weibull, Pareto, Kumaraswamy, lognormal, Gompertz etc. have been generalized using various methods of generating distribution. Shaw and Buckley (2007) introduced an interesting method of adding new parameter to the existing one using the quadratic rank transmutation map (QRTM) in order to generate a flexible family of distributions. A random variable X is said to have a transmuted distribution if its distribution function is given by

$$F(x) = (1 + \lambda)G(x) - \lambda(G(x))^2 \tag{1}$$

where $G(x)$ denotes cumulative distribution function of the baseline distribution and $\lambda \in [-1, 1]$ is

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the transmuting parameter and

$$f(x) = g(x)\left(1 + \lambda - 2\lambda(G(x))\right) \tag{2}$$

is the corresponding probability density function of the family distribution, where $g(x)$ is the probability density function of the baseline distribution. A significant amount of work has been done towards developing a new transmuted model and several of them have been investigated. Aryal and Tsokos (2011) proposed transmuted Weibull, a generalization of the Weibull probability distribution, Khan and King (2013) proposed transmuted modified Weibull distribution, a generalization of modified Weibull distribution. Elbatal (2013) studied transmuted modified inverse Weibull distribution, Elbatal and Aryal (2013) worked on the transmuted additive Weibull distribution. Merovci (2013,2014) introduced transmuted Rayleigh and later developed transmuted generalized Rayleigh distributions and their properties. Also, Merovci and Elbatal (2014a) introduced transmuted Lindley Geometric distribution using quadratic rank transmutation map and studied several properties of the distribution. Mohammad (2016) studied transmuted exponentiated inverse Rayleigh distribution and its applications. Khan et al. (2016) proposed transmuted new generalized Weibull distribution for life time modelling.

The Weibull-Rayleigh distribution introduced by Merovci and Elbatal (2015), has the cumulative distribution function (cdf)

$$G(x,\alpha,\beta,\theta) = 1 - exp\left(-\alpha exp\left(\frac{\theta x^2}{2} - 1\right)\right)^{\beta}; x > 0, \alpha, \beta, \theta > 0 \tag{3}$$

and the probability density function (pdf)

$$g(x,\alpha,\beta,\theta) = \alpha\beta\theta x exp\left(\frac{\theta x^2}{2}\right) exp\left(\frac{\theta x^2}{2} - 1\right)^{\beta-1} exp\left(-\alpha exp\left(\frac{\theta x^2}{2} - 1\right)\right)^{\beta} \tag{4}$$

$$; x > 0, \alpha, \beta, \theta > 0$$

The rest of the paper is structured as follows: Section 2 discusses the probability density function and the cumulative distribution function of TWR distribution. In Section 3 some properties of the new distribution are derived. The estimation of parameters using maximum likelihood estimation (MLE) and application of the proposed model to real life data sets are provided in section 4. Lastly in section 5, we make some useful conclusion.

2. Methodology

2.1 *Transmuted Weibull-Rayleigh Distribution*

Transmuted Weibull Rayleigh (TWR) distribution is a four parameter distribution derived by inserting equation (3) and (4) in equation (1) and (2). Thus the cdf of Transmuted Weibull Rayleigh is given by

$$F(x,\alpha,\beta,\theta,\lambda) = \left(1 - \exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right) - 1\right)^{\beta}\right)\right)\left(1 + \lambda\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right) - 1\right)^{\beta}\right)\right)$$

$$= 1 - (1-\lambda)\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right) - 1\right)^{\beta}\right) - \lambda\exp\left(-2\alpha\left(\exp\left(\frac{\theta x^2}{2}\right) - 1\right)^{\beta}\right) \tag{5}$$

$$; x > 0, \alpha, \beta, \theta > 0, |\lambda| > 1$$

And the probability density function is given by

$$f(x,\alpha,\beta,\theta,\lambda)=\alpha\beta\theta x\exp\left(\frac{\theta x^2}{2}\right)\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta-1}\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)$$

$$\times\left[1+\lambda-2\lambda\left(1-\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\right)\right] \tag{6}$$

$$;x>0,\alpha,\beta,\theta>0,|\lambda|>1$$

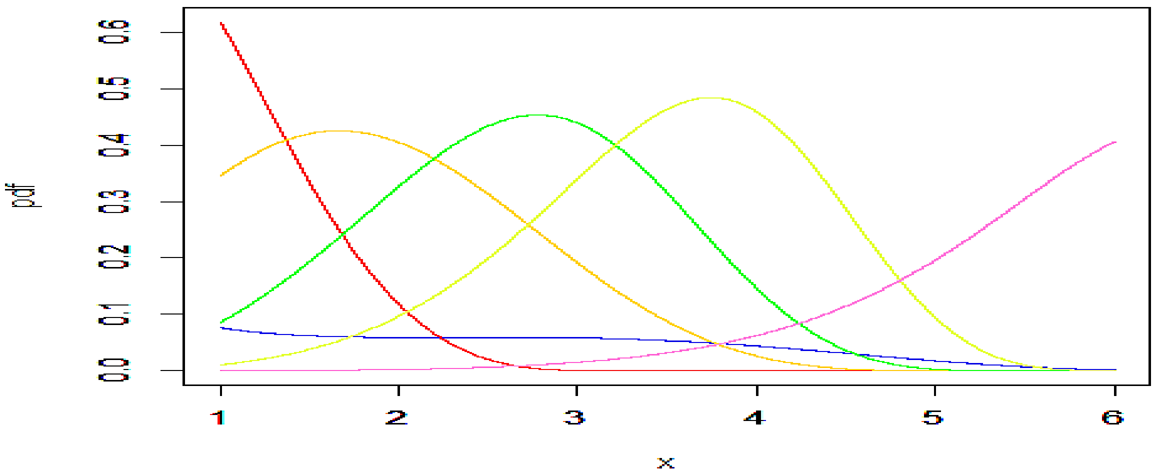

Figure 1. Pdf plot of TWR

Figure 1 illustrates some of the possible shapes of the pdf of TWR distribution for selected values of the parameters $\alpha = 0.2; 0.4; 0.6; 0.8; 1; 2\beta = 0.1; 0.8; 1; 1.5; 2; 2.5, \theta = 1; 1.5; 2.5; 3; 3.5; 4.5, \lambda = 0.5$ with colour shapes blue, red, orange, green, greenyellow and hotpink respectively.

3. Properties

In this section, we study some statistical properties of TWR distribution, comprising reliability analysis, moments, order statistics, quantiles and random number generation.

3.1 *Reliability Analysis*

The reliability function of the distribution is given by

$$R(x)=1-F(x)$$

Therefore, the reliability function for TWR distribution is simplified to give

$$R(x)=(1-\lambda)\exp\left(-\alpha\left(\exp(\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)+\lambda\exp\left(-2\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right) \tag{7}$$

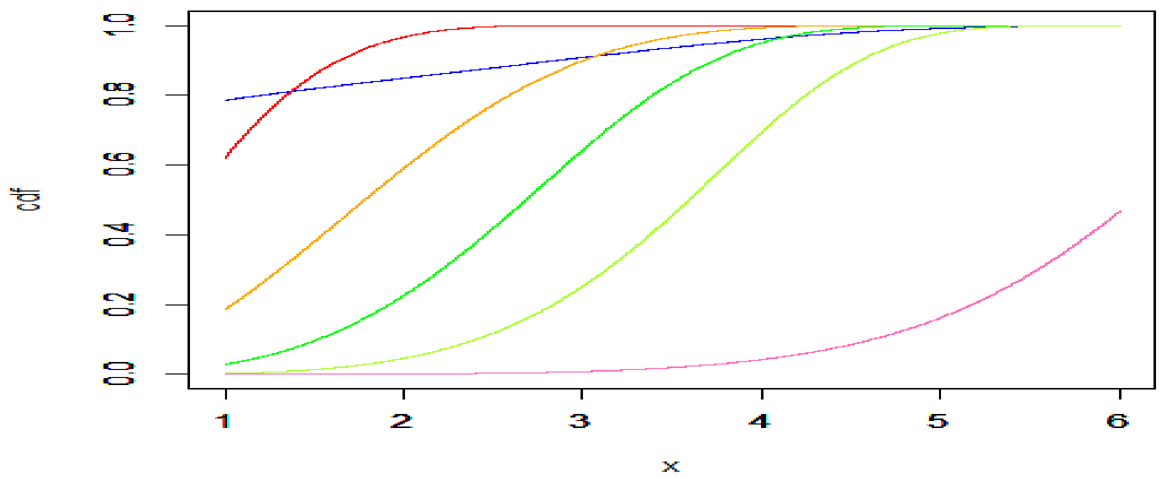

Figure 2. Cdf plot of TWR

Figure 2 illustrates some of the possible shapes of the cdf of TWR distribution for selected values of the parameters $\alpha = 0.2; 0.4; 0.6; 0.8; 1; 2\beta = 0.1; 0.8; 1; 1.5; 2; 2.5, \theta = 1; 1.5; 2.5; 3; 3.5; 4.5, \lambda = 0.5$ with colour shapes blue, red, orange, green, greenyellow and hotpink respectively.

and the hazard function, mathematically given by

$$h(x) = \frac{f(x)}{R(x)}$$

hence, the hazard function(failure rate) of the TWR distribution is

$$h(x) = \frac{\alpha\beta\theta x \exp\left(\frac{\theta x^2}{2}\right)\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta-1}\left[1+\lambda-2\lambda\left(1-\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\right)\right]}{\left((1-\lambda)-\lambda\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\right)} \tag{8}$$

3.2 r^{th} *Moments*

$$E(x^r) = \int_0^\infty x^r f(x)dx = \int_0^\infty x^r \alpha\beta\theta x \exp\left(\frac{\theta x^2}{2}\right)\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta-1} \exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)$$

$$\times\left[1+\lambda-2\lambda\left(1-\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\right)\right] dx; \qquad x > 0, \alpha, \beta, \theta > 0, |\lambda| > 1$$

$$= \alpha\beta\theta(1-\lambda)\int_0^\infty x^{r+1}\exp\left(\frac{\theta x^2}{2}\right) - \left(\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta-1} dx$$

$$+2\alpha\beta\theta\lambda\int_0^\infty x^{r+1}\exp\left(\frac{\theta x^2}{2}\right) - 2\left(\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta-1} dx \tag{9}$$

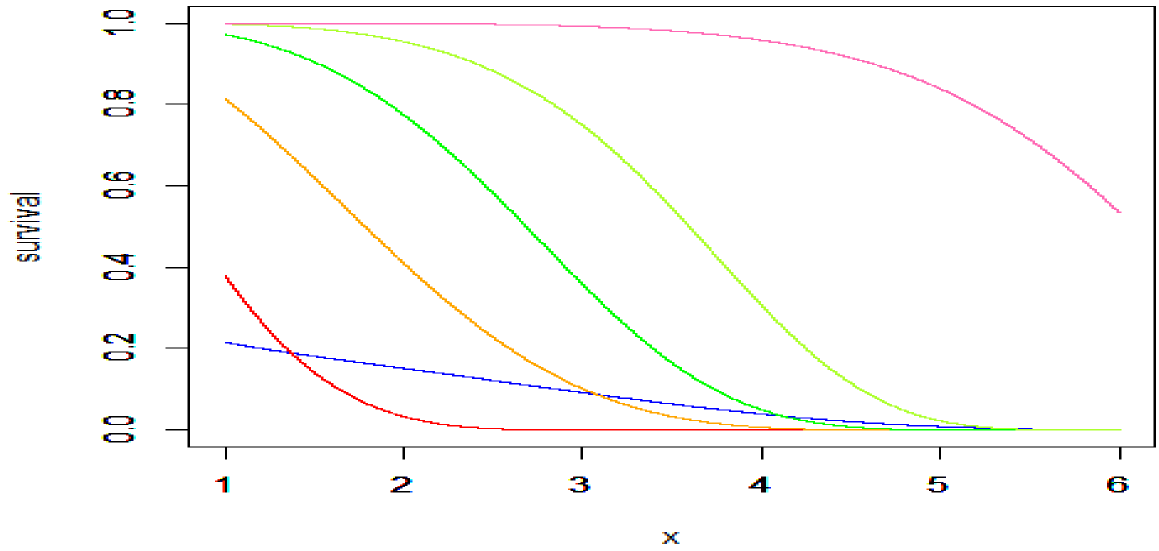

Figure 3. Survival function plot of TWR

Figure 3 illustrates some of the possible shapes of the survival function of TWR distribution for selected values of the parameters $\alpha = 0.2; 0.4; 0.6; 0.8; 1; 2 \beta = 0.1; 0.8; 1; 1.5; 2; 2.5, \theta = 1; 1.5; 2.5; 3; 3.5; 4.5, \lambda = 0.5$ with colour shapes blue, red, orange, green, greenyellow and hotpink respectively.It was observed that the survival function is decreasing depending on the values of the parameters.

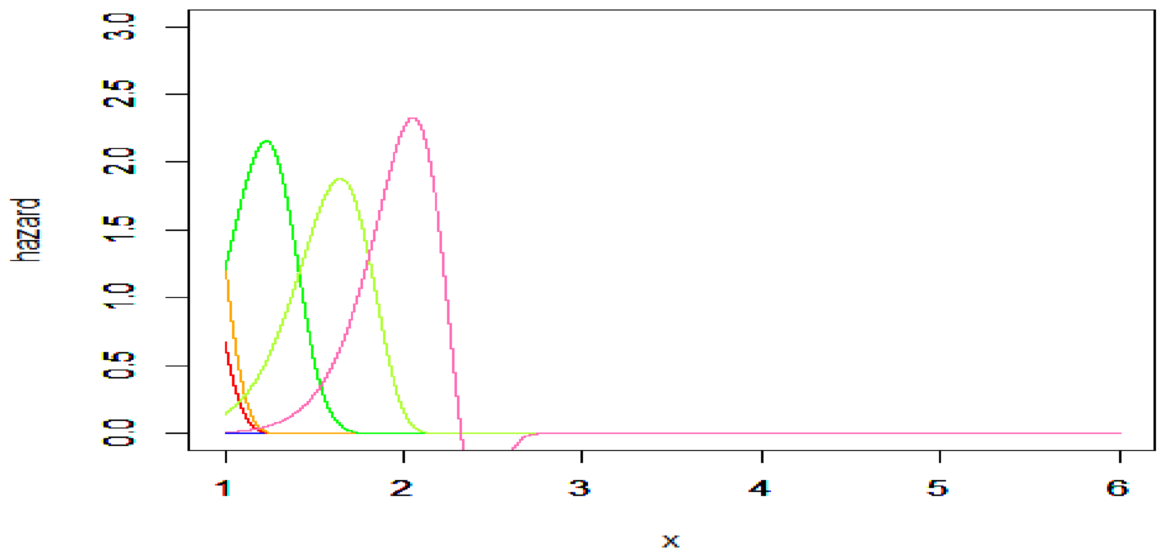

Figure 4. Survival function plot of TWR

Figure 4 illustrates some of the possible shapes of the hazard function of TWR distribution for selected values of the parameters $\alpha = 0.2; 0.4; 0.6; 0.8; 1; 2 \beta = 0.1; 0.8; 1; 1.5; 2; 2.5, \theta = 1; 1.5; 2.5; 3; 3.5; 4.5, \lambda = 0.5$ with colour shapes blue, red, orange, green, greenyellow and hotpink respectively.

The mean and variance of TWR can be found analytically using the 1^{st} and the 2^{nd} moments.

3.3 *Quantiles and Random Number Generation*

The quantiles of a mathematically specified distribution say the jth quantile, is derived by solving the equation

$$U = F(x) = (1+\lambda)G(x) - \lambda\left(G(x)\right)^2$$

In the equation $U = F(x) = (1+\lambda)G(x) - \lambda\left(G(x)\right)^2$, $U \in (0,1)$ and $G(x)$ the cdf of the baseline distribution.

$$\lambda\left(G(x)\right)^2 - (1+\lambda)G(x) + u = 0$$

$$G(x) = \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4u\lambda}}{2\lambda}$$

$$1-\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right) = \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4u\lambda}}{2\lambda}$$

$$\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right) = 1-\left(\frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4u\lambda}}{2\lambda}\right)$$

$$-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta} = \ln\left(\frac{(\lambda-1) \pm \sqrt{(1+\lambda)^2 - 4u\lambda}}{2\lambda}\right)$$

$$\exp\left(\frac{\theta x^2}{2}\right) = 1-\left[\frac{1}{\alpha}\ln\left(\frac{(\lambda-1) \pm \sqrt{(1+\lambda)^2 - 4u\lambda}}{2\lambda}\right)\right]^{\frac{1}{\beta}}$$

Taking logarithm of both sides

$$\frac{\theta x^2}{2} = \ln\left[1-\left(\frac{1}{\alpha}\ln\left(\frac{(\lambda-1) \pm \sqrt{(1+\lambda)^2 - 4u\lambda}}{2\lambda}\right)\right)^{\frac{1}{\beta}}\right]$$

Hence, the quantile function is

$$Q(u) = F^{-1}(u) = \sqrt{\frac{2}{\theta}\ln\left[1-\left(\frac{1}{\alpha}\ln\left(\frac{(\lambda-1) \pm \sqrt{(1+\lambda)^2 - 4u\lambda}}{2\lambda}\right)\right)^{\frac{1}{\beta}}\right]} \tag{10}$$

Once a random number,U, is available which after scaling may be regarded as uniformly distributed on $(0,1)$, we can generate a random value for any distribution using $X_u = F^{-1}(u)$ which implies

$U = F(x)$ The random variable X from TWR distribution is given by

$$X_u = \sqrt{\frac{2}{\theta}\ln\left[1 - \left(\frac{1}{\alpha}\ln\left(\frac{(\lambda-1) \pm \sqrt{(1+\lambda)^2 - 4u\lambda}}{2\lambda}\right)\right)^{\frac{1}{\beta}}\right]}; for\lambda \neq 0 \tag{11}$$

Setting $u = \frac{1}{2}$ in $eq.$12, it follows that the median M of X is

$$M = X_{0.5} = \sqrt{\frac{2}{\theta}\ln\left[1 - \left(\frac{1}{\alpha}\ln\left(\frac{(\lambda-1) \pm \sqrt{1+\lambda^2}}{2\lambda}\right)\right)^{\frac{1}{\beta}}\right]} \tag{12}$$

The lower and the upper quartiles can also be derived from $eq.$12 at $u = 0.25$ and $u = 0.75$ respectively.

The Bowley's and Kelly's formula for finding coefficient of skewness are respectively given by

$$S_{k(B)} = \frac{x_{0.75} - 2x_{0.5} + x_{0.25}}{x_{0.75} - x_{0.25}} \tag{13}$$

$$S_{k(k)} = \frac{x_{0.9} - 2x_{0.5} + x_{0.1}}{x_{0.9} - x_{0.1}} \tag{14}$$

And the Moors' formula for finding coefficient of kurtosis is given by

$$K_{(u)} = \frac{x_{0.875} - x_{0.625} - x_{0.375} + x_{0.125}}{x_{0.75} - x_{0.25}} \tag{15}$$

3.4 *Order Statistics, Estimation of Parameters of TWR*

3.4.1 Order Statistics

Let $X_1, X_2, X_3, \ldots, X_n$ be a simple random sample of size n from the TWR distribution with the pdf and cdf given by $eq.$5 and $eq.$6 respectively. Let $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$ be the order statistics from the sample, the distribution of the order statistics is given by;

$$f_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\left[F(x)\right]^{r-1}\left[1 - F(x)\right]^{n-r} f(x)$$

for $r = 1, 2, 3, ..., n$. Therefore, the distribution of the order statistics of TWR is given by

$$f_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!}$$

$$\times\left[\left(1 - \exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right) - 1\right)^{\beta}\right)\right)\left(1 + \lambda\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right) - 1\right)^{\beta}\right)\right)\right]^{r-1}$$

$$\times\left[(1-\lambda)\exp\left(-\alpha\left(\exp(\left(\frac{\theta x^2}{2}\right) - 1\right)^{\beta}\right) + \lambda\exp\left(-2\alpha\left(\exp\left(\frac{\theta x^2}{2}\right) - 1\right)^{\beta}\right)\right]^{n-r}$$

$$\times \alpha\beta\theta x \exp\left(\frac{\theta x^2}{2}\right)\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta-1}\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)$$

$$\times\left[1+\lambda-2\lambda\left(1-\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\right)\right] \tag{16}$$

The pdf of the smallest and largest order statistics are $F_{(1)}(x)$ and $F_{(n)}(x)$.

3.4.2 Estimation of Parameters of TWR

The method of maximum likelihood estimation (MLE) is used to estimate the parameters of the TWR distribution. Let $X_1, X_2, X_3, \ldots, X_n$ be a sample of size n from the TWR distribution, the likelihood function is given by

$$L(X_1, X_2X_3, \ldots, X_n|\alpha,\beta,\theta,\lambda) = \prod_{i=1}^{n}\alpha\beta\theta x \exp\left(\frac{\theta x^2}{2}\right)\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta-1}$$

$$\times\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\times\left[1+\lambda-2\lambda\left(1-\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\right)\right]$$

$$\log L = n\log(\alpha\beta\theta)+\sum_{i=1}^{n}\log(x_i)+\frac{\theta}{2}\sum_{i=1}^{n}(x_i)^2+(\beta-1)\sum_{i=1}^{n}\left(\log\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)\right)$$

$$-\alpha\sum_{i=1}^{n}\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}+\sum_{i=1}^{n}\log\left[1-\lambda+2\lambda\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right] \tag{17}$$

Let $l = LogL$. Taking the derivative of Log L with respect to each parameter α, β, θ and λ and setting the result equals to zero, we obtain maximum likelihood estimates. Hence, the partial derivative of $LogL$ with respect to each parameter is given by

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha}-\sum_{i=1}^{n}\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}+2\lambda\sum_{i=1}^{n}\frac{\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\exp\left(-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)}{\left[1-\lambda+2\lambda\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right]} \tag{18}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\alpha}+\sum_{i=1}^{n}\left(\log\exp\left(\frac{\theta x^2}{2}\right)-1\right)-\alpha\sum_{i=1}^{n}\left(\log\exp\left(\frac{\theta x^2}{2}\right)-1\right)\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}$$

$$+\sum_{i=1}^{n}\frac{2\alpha\lambda\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\sum_{i=1}^{n}\left(\log\exp\left(\frac{\theta x^2}{2}\right)-1\right)}{1-\lambda+2\lambda\left(\exp\left(-\alpha\left(\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\right)\right)} \tag{19}$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta}+\frac{1}{2}\sum_{i=1}^{n}x_i^2+(\beta-1)\sum_{i=1}^{n}\frac{1}{2}x_i^2\frac{\exp\left(\frac{\theta x^2}{2}\right)}{\exp\left(\frac{\theta x^2}{2}\right)-1}-\frac{1}{2}\alpha\beta\sum_{i=1}^{n}x_i^2\exp\left(\frac{\theta x^2}{2}\right)\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta-1}$$

$$-\sum_{i=1}^{n}\frac{\alpha\beta\sum_{i=1}^{n}x_i^2\exp\left(\frac{\theta x^2}{2}\right)\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta-1}\left(\exp\left(-\alpha\left(\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\right)\right)}{1-\lambda+2\lambda\left(\exp\left(-\alpha\left(\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\right)\right)} \tag{20}$$

$$\frac{\partial\ell}{\partial\lambda}=\sum_{i=1}^{n}\frac{-1-2\left(\exp\left(-\alpha\left(\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\right)\right)}{1-\lambda+2\lambda\left(\exp\left(-\alpha\left(\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}\right)\right)\right)} \tag{21}$$

Newton-Raphson maximisation numerical technique is used to solve these equations, since there is no closed form solution to them. If all the second order derivatives exist, we obtain a 4×4 inverse dispersion matrix given by

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\theta}\\ \hat{\lambda}\end{pmatrix}=\mathbf{N}\left[\begin{pmatrix}\alpha\\ \beta\\ \theta\\ \lambda\end{pmatrix},\begin{pmatrix}V_{\alpha\alpha} & V_{\alpha\beta} & V_{\alpha\theta} & V_{\alpha\lambda}\\ V_{\beta\alpha} & V_{\beta\beta} & V_{\beta\theta} & V_{\beta\lambda}\\ V_{\theta\alpha} & V_{\theta\beta} & V_{\theta\theta} & V_{\theta\lambda}\\ V_{\lambda\alpha} & V_{\lambda\beta} & V_{\lambda\theta} & V_{\lambda\lambda}\end{pmatrix}\right]$$

$$\mathbf{V^{-1}}=-\mathbf{E}\begin{bmatrix}V_{\alpha\alpha} & V_{\alpha\beta} & V_{\alpha\theta} & V_{\alpha\lambda}\\ V_{\beta\alpha} & V_{\beta\beta} & V_{\beta\theta} & V_{\beta\lambda}\\ V_{\theta\alpha} & V_{\theta\beta} & V_{\theta\theta} & V_{\theta\lambda}\\ V_{\lambda\alpha} & V_{\lambda\beta} & V_{\lambda\theta} & V_{\lambda\lambda}\end{bmatrix} \tag{22}$$

where $V_{\alpha\alpha}=\frac{\partial^2\ell}{\partial\alpha^2}, V_{\alpha\beta}=\frac{\partial^2\ell}{\partial\alpha\beta}, V_{\alpha\theta}=\frac{\partial^2\ell}{\partial\alpha\theta}, V_{\alpha\lambda}=\frac{\partial^2\ell}{\partial\alpha\lambda}, V_{\beta\alpha}=\frac{\partial^2\ell}{\partial\beta\alpha}, V_{\beta\beta}=\frac{\partial^2\ell}{\partial\beta^2}, V_{\beta\theta}=\frac{\partial^2\ell}{\partial\beta\theta}, V_{\beta\lambda}=\frac{\partial^2\ell}{\partial\beta\lambda}, V_{\theta\alpha}=\frac{\partial^2\ell}{\partial\theta\alpha}, V_{\theta\beta}=\frac{\partial^2\ell}{\partial\theta\beta}, V_{\theta\theta}=\frac{\partial^2\ell}{\partial\theta^2}, V_{\theta\lambda}=\frac{\partial^2\ell}{\partial\theta\lambda}, V_{\lambda\alpha}=\frac{\partial^2\ell}{\partial\lambda\alpha}, V_{\lambda\beta}=\frac{\partial^2\ell}{\partial\lambda\beta}, V_{\lambda\theta}=\frac{\partial^2\ell}{\partial\lambda\theta}, V_{\lambda\lambda}=\frac{\partial^2\ell}{\partial\lambda^2}$.

The solution of the inverse matrix of the observed information matrix in *eq*.19 gives the asymptotic variance and co-variance of the maximum likelihood estimators $\hat{\alpha}$, $\hat{\beta}$, $\hat{\theta}$ and $\hat{\lambda}$. Approximate $100(1-\gamma)$ percent confidence intervals for α, β, θ and λ are given respectively by

$$\hat{\alpha}\pm Z_{\frac{\gamma}{2}}\sqrt{V_{\alpha\alpha}}, \hat{\beta}\pm Z_{\frac{\gamma}{2}}\sqrt{V_{\beta\beta}}, \hat{\theta}\pm Z_{\frac{\gamma}{2}}\sqrt{V_{\theta\theta}}, \hat{\lambda}\pm Z_{\frac{\gamma}{2}}\sqrt{V_{\lambda\lambda}} \tag{23}$$

Where Z_γ is the γ^{th} percentile of the standard normal distribution.

4. Application

In this section, we compare the result of fitting the Transmuted Weibull Rayleigh distribution, Weibull Rayleigh distribution, Weibull distribution and Rayleigh distribution to a set of data extracted from (Merovci and Elbatal, 2015) and previously been studied by Meeker and Escobar (1998). The analyses were done with the aid of R software. The data are as follows;

2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00,
3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66

4.1 *Results of Data Analysis*

To compare the distributions, we consider criteria like $-2l$, AIC (Akaike information criterion) and AICC (Corrected Akaike information criterion) for the data set. The model corresponding to the

Table 1. Summary of data set

Min.	Q_1	Q_2	Q_3	Mean	Max.	Variance	Skewness	Kurtosis
0.0200	0.6875	1.9650	2.9820	1.7700	3.0000	1.3223	-0.2840	1.4536

Table 2. The ML estimates, standard error, log-likelihood, AIC and AICC for data set

Model	ML Est.	St. Err.	-2LL	AIC	AICC
5*TWR	$\hat{\alpha} = 0.000254$	0.00023	10.6290	16.6289	18.2289
	$\hat{\beta} = 0.01032$	NA			
	$\hat{\theta} = 3.2000$	6.1205			
	$\hat{\lambda} = 0.50000$				
5*Weibull Rayleigh	$\hat{\alpha} = 0.6325$	0.1022	92.3174	98.3175	99.6622
	$\hat{\beta} = 0.3995$	7.9912			
	$\hat{\theta} = 0.1129$	2.2594			
5*l Weibull	$\hat{\alpha} = 1.8802$	0.2821	92.32	96.32	96.76
	$\hat{\beta} = 1.2650$	0.2044			
Rayleigh	$\hat{\alpha} = 0.4533$	0.0828	101.77	103.77	103

lowest $-2l$, AIC and AICC values is regarded as the best model.

$$AIC = 2k - 2l$$

$$AICC = AIC + \frac{2k(k+1)}{n-k-1} \tag{24}$$

where k is the number of parameters in the statistical model, n is the sample size and l is the maximized value of the log-likelihood function under the considered model.

Table 3. Survival Analysis of Transmitted Weibull Rayleigh distribution using 0.6671

TIME (x)	$S(x)$	Number
0.5	0.8056252	24
1.0	0.5573718	17
1.5	0.3207426	10
2.0	0.13564	4
2.5	0.03358015	1

Table 4. Hazard Function of TWR distribution using TWR data

TIME (x)	$H(x)$
0.5	0.5956897
1.0	0.8931025
1.5	1.357844
2.0	2.157174
2.5	3.574922

5. Discussion

Figure 1 and 4 show the probability density and hazard function which take many forms such as symmetric, asymmetric, unimodal and bath rub depending on the values of the parameters. In Figure

2, the cumulative distribution function is on increase based on the values of the parameters while in Figure 3 the survival function decreases depending on the values of the parameter. Table 2 shows the values of MLEs, Standard error, $-2LL$, AIC and AICC. The model with the lowest $-2LL$, AIC and AICC is regarded as the best model. Tables 3 & 4 show the reliability analysis of the TWR distribution. The Table 3 and 4 indicate as the time rate increases the survival rate drops. Also as the time rate increases so the hazard rate increases. The Transmuted Weibull Rayleigh distribution is the best among other distributions used for fitting the data set.

6. Conclusion

In this article, a new Transmuted Weibull Rayleigh distribution which is an extension of the Weibull Rayleigh distribution, which is more flexible in modelling real data, is proposed. The graph of the probability density and the hazard functions show that the TWR distribution can model different data set. We derive the reliability functions, moments, quantile and order statistics of the distribution. The estimation of parameters is done by the method of maximum likelihood estimation. The Transmuted Weibull-Rayleigh distribution appeared to be the best out of the Weibull-Rayleigh distribution, the Weibull distribution and the Rayleigh distribution when applied to data set.

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Appendix

R-codes used for MLE

```
# Transmuted Weibull-Rayleigh#
library(maxLik)
library(miscTools)
loglikTWR< −function(p){
a< −p[1]; b< −p[2]; c< −p[3]; t< −.5
-nblog(a)+nlog(b)-nlog(c)+sum(log(x))+(1/(2c))sum(x∧2)
+(b-1)sum(log((exp(x∧2/(2c))-1)))-(a∧-b)sum((exp(x∧2/
(2c))-1)∧b) +sum(log(1-t+2texp(-(a∧-b)exp((x∧2/(2c))-1)∧b)))
d< − maxLik::maxLik(loglikTWR,start=c(.00051,.02,3.2))
summary(d)
AIC(d)
vcov(d)
# Weibull-Rayleigh#
library(Newdistns)
mweibullg(”rayleigh”, data=x, starts=c(1,1,1),
method = ”BFGS” )
Weibull
library(fitdistrplus)
fwmle < − fitdist(x, ”weibull”, method=”mle”)
summary(fwmle)
gofstat(fwmle)
# Rayleigh #
library(VGAM)
library(fitdistrplus)
fRmle < − fitdist(x, drayleigh, method=”mle”,start= alpha.est)
summary(fRmle)
gofstat(fRmle)
```