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Personnel Movement On a Fixed Size Multi-Step G-Grade Manpower System

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Abstract. This paper considers aspect of a network of personnel flow in a fixed size hierarchically structured Markov manpower system under intra/inter-grade mobility-specific feature. The study shows that as long as the Markov chain describing the system is stochastic and irreducible, the system remains finite and bounded. In the case study, the structure of the University system under a fixed size manpower policy would result to a bottom-heavy structure. This work has revisited the use of Markov chain as a tool to describing the evolution of a hierarchical manpower. The notion of intra/inter-grade mobility in a fixed size manpower system has been investigated. In such a system, the manpower structure was finite and bounded as the transition matrix of the system was stochastic and irreducible. An illustrative example was sourced using practical data from a private university in Nigeria to illustrate the new method. The real life example demonstrates that the technique of attributing the same promotion probability to each member of the same grade can be advanced further to the grade step. Also, the structure of the system was extrapolated in order to ascertain whether the fixed size policy would result to a bottom-heavy or top-heavy structure.

Keywords: Hierarchically structured system, Intra/inter-grade mobility, Manpower system, Markov chain, Stochastic matrix

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1. Introduction

Individuals of different cohorts in an organisation (such as the civil service, university, etc.) usually move through a network of categories defined by the grade levels and the grade steps in an organisation. These movements may also be referred to as horizontal mobility (i.e., the intra-grade movement between

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grade steps) and vertical mobility (i.e., the inter-grade movement between grade levels).
These common realistic movements in organisations motivate this study. It is the aim of this study to investigate the evolution of manpower structure under horizontal and vertical mobility for a fixed size system. The fixed size system is appropriate in practice for a system with limited skilled personnel availability, facility and budget restrictions (cf. Tsaklidis, 1996; Komarudin et al., 2015). Manpower system consists of personnel working together for the purpose of achieving the common goal of an organization.
In most manpower models, such as: (Amenaghawon et al, 2020a; Ezugwu and Igbinosun 2020; Vassiliou 2021), the manpower system is hierarchically graded into mutually exclusive and exhaustive grades so that each member of the system may belong to one and only one of the grades at any given times.
In the Markovian approach for modelling manpower systems, Markov models can be classified into homogeneous or nonhomogeneous, based on the nature of the system's dependency on time. A Markov chain model is said to be homogeneous if the transition probabilities of the members are assumed to be independent of time (that is, they do not change over time). Examples of such models include; (Ekhosuehi and Osagiede, 2006; Ekhosuehi et al, 2017; Ezugwu and Ologun, 2017; Ezugwu and Igbinosun , 2020).
A Markov chain model is said to be non-homogeneous if the transition probabilities of the members are assumed to be dependent on time. Examples include (Vassiliou, 2021; Vassiliou, 2022). The work of (Ezugwu et al, 2024) considers a hierarchically non-homogeneous manpower system in which promotion of employees is only assessable based on the level of innovativeness and job performance capability. (Amenaghawon et al, 2020b), reviews the manpower planning literature with specific interest on manpower systems modelled within the markov chain context and highlighted the methodological issues arising from varying the unit interval of the markov manpower system in discrete time. (Carette and Guerry 2023) considers the problem of finding the transition rates of a continuous-time homogeneous markov chain under the empirical condition that the state changes at most once during a time interval of unit length.
It is desirable to define the states of a system to allow for variations due to the intra-grade and the inter-grade features of the system (Bartholomew et al., 1991). In this sense, the population of the manpower system may be stratified into G grade levels and the personnel of the gth grade, $g = 1, 2, \wedge, G$, can be classified into $k_{(g)}$ homogeneous, mutually exclusive and exhaustive states, denoted by the set $S_{(g)} = 1_{(g)}, 2_{(g)}, \wedge, k_{(g)}$. In most studies on mathematical human resources planning only intergrade mobility is considered (cf. TsklidIs, 1996; Komarudin et al., 2015), that is $k_{(g)} = 1$ for each $g = 1, 2, \wedge, G$, and we speak of a G-grade manpower system (Ekhosuehi, 2021a,b; Guerry, 2021; Vassiliou, 2021).
In the present study, $k_{(g)} > 1$ for at least one g and so we talk of a multi-step G-grade manpower system. It is more realistic to unbundle the steps in a grade of a hierarchical manpower system, rather than ignoring it. This is because the grade step tends to indicate the level of seniority in a grade. This study relies on Markov census analysis as the manpower system is made up of members of different cohorts wherein staff members can leave the system at any stage

and others will join the system. The study deals with transversal data on the different cohorts. The rest of this paper is structured into five sections. The second section contains model description. The theoretical results of this work are contain in section three, while section four contains the numerical results and a brief discussion, the concluding remarks are given in the last section.

2. Materials and Method

Consider a manpower system stratified into G categories according to the grade levels. Let $S_{(g)}$ be the set of categories the personnel in a grade g, $g = 1, 2, \wedge, G$, can be classified. Suppose that it is possible to move from one step to the next higher step of the same grade (intra-mobility) and that transitions can occur between the grades of the system (inter-mobility). Let the structure of the system at any period of time t be denoted by $\sum_{g-1}^{G} k_{(g)}$ -tuple stock vector. $(n_1(t)|n_2(t)| \wedge |n_g(t))$.
The vector $n_{(g)(t)} = (n_1(t), n_2(t), \wedge, n_{k_{(g)}}(t))$ is the personnel distribution in grade g. When the transition probabilities are stationary, then the stock $n_i(t), i = 1, 2, \wedge, k_{(g)}$, changes after a period of time in such fashion that: a proportion $p_{ii}^{(g)}$ is in step i of grade level g, a proportion $p_{ij}^{(g)}$ moves to the next higher step $j = i + 1 \in S_g$ of grade level g, a proportion $p_{(i)h}^{(g)}$ moves from step i of grade level g to another step of grade level $h \neq g$, a proportion $w_i^{(g)}$ leaves the system in step i of grade level g, or a proportion of recruits, r_i^g, enter step i of grade level g. Thus, transitions can occur from and towards the external environment (recruitment and turnover flows).
Assume a fixed size manpower system. In such a system, the total number of personnel is known and there is no need to investigate the notion of downsizing in order to achieve economies of scale. Let the $1 \times \sum_{g-1}^{G} k_{(g)}$ vector w denote the vector of wastage for whatever reasons in the system and the $\sum_{g-1}^{G} k_{(g)} \times 1$ vector r represent the vector of recruits. Let matrices, $\Phi^g_{k_{(g)} \times k_{(g)}}, g = 1, 2, \wedge, G$, define the intra-grade mobility. In these matrices, there are two possible transitions for an individual staff: either the individual leaves the grade or increases the step in the grade by one unit in the grade. Thus, the rows of $\Phi^g_{k_{(g)} \times k_{(g)}}$ may not all add up to one. Therefore, $\Phi^g_{k_{(g)} \times k_{(g)}}$ is sub-stochastic. Let $\Theta^{g+h}_{k_{(g)} \times k_{(g+h)}}, g = 1, 2, \wedge, G, g \neq h$, denote the matrix of inter-grade mobility in the system. For instances where movements are not possible between the grades g and $g + h$, $\Theta^{g+h}_{k_{(g)} \times k_{(g+h)}}$ is a null matrix. The matrices $\Phi^g_{k_{(g)} \times k_{(g)}}$ and $\Theta^{g+h}_{k_{(g)} \times k_{(g+h)}}$ can be put together to form a $(\sum_{g-1}^{G} k_{(g)} \times \sum_{g-1}^{G} k_{(g)})$ block matrix, P, of the

internal transitions in the system as

$$P = \begin{bmatrix} \Phi^1_{k_{(1)}\times k_{(1)}} & \theta^2_{k_{(1)}\times k_{(2)}} & \theta^3_{k_{(1)}\times k_{(3)}} & \wedge & \theta^G_{k_{(1)}\times k_{(G)}} \\ \theta^1_{k_{(2)}\times k_{(1)}} & \Phi^2_{k_{(2)}\times k_{(2)}} & \theta^3_{k_{(2)}\times k_{(3)}} & \wedge & \theta^G_{k_{(2)}\times k_{(G)}} \\ \theta^1_{k_{(3)}\times k_{(1)}} & \theta^2_{k_{(3)}\times k_{(2)}} & \Phi^3_{k_{(3)}\times k_{(3)}} & \wedge & \theta^G_{k_{(3)}\times k_{(G)}} \\ M & M & M & 0 & M \\ \theta^1_{k_{(G)}\times k_{(1)}} & \theta^2_{k_{(G)}\times k_{(2)}} & \theta^3_{k_{(G)}\times k_{(3)}} & \wedge & \Phi^G_{k_{(G)}\times k_{(G)}} \end{bmatrix} \tag{1}$$

Suppose that leavers from each grade g are immediately replaced by new recruits. Then we can represent the Markov chain describing the consequential effect of recruitment to replace wastage as $Q = P + w'r$, where $w'r$ is the consequential effect of recruitment to replace wastage. For the system at hand, for the homogeneous Markov system with seniority-specific levels (defined by the grade steps), the expected structure is a special form of the evolution of manpower structures (Bartholomew et al., 1991). The first-order difference equation describing the dynamics of the system is given as

$$(n_{(1)}(t+1)(|n_{(2)}(t+1)| \wedge |(n_{(7)}(t+1))| = (n_{(1)}(t)|n_{(2)}(t)| \ldots |n_{(7)}(t)|)Q \tag{2}$$

where $n_{(g)}(t) = [n_1(t), n_2(t), \wedge, n_{k_g}(t)]$ is the distribution of staff in grade g at time t . The notation $k_{(g)}$ is the number of steps in a grade g according to the Consolidated University Academic Staff Salary (CONUASS) structure: $k_{(1)} = 6; k_{(2)} = 8; k_{(3)} = 8; k_{(4)} = 9; k_{(5)} = 13; k_{(6)} = 10; k_{(7)} = 10.$

3. Results and Discussion

3.1 *Theoretical results*

We present two main theoretical results. The first is on the limiting value of Q and the second is centred on the limiting structure of the system.

Proposition 1: The matrix Q has a limiting value provided it is irreducible.

Proof

Let Q be $(\sum_{g-1}^{G} k_{(g)} \times \sum_{g-1}^{G} k_{(g)})$ irreducible stochastic matrix describing transitions in the system. With $(n_{(1)}(t+1)(|n_{(2)}(t+1)| \wedge |(n_{(G)}(t+1))| = (n_{(1)}(t)|n_{(2)}(t)| \wedge |n_{(G)}(t)|)Q$, the generation function vector $g(z)$, which is associated with $(n_{(1)}(t)|n_{(2)}(t)| \wedge |n_{(G)}(t)|)$, is defined as

$$g(z) = \sum_{t-0}^{\infty} (n_{(1)}(t)|n_{(2)}(t)| \wedge |n_{(G)}(t)|)z^t \tag{3}$$

It follows that

$$g(z)Q = \sum_{t-0}^{\infty}(n_{(1)}(t)|n_{(2)}(t)| \wedge |n_{(G)}(t)|)Qz^t = \sum_{t-0}^{\infty}(n_{(1)}(t+1)|n_{(2)}(t+1)| \wedge |n_{(G)}(t+1)|)z^t$$
$$= \frac{1}{z}\sum_{t-0}^{\infty}(n_{(1)}(t+1)|n_{(2)}(t+1)| \wedge |n_{(G)}(t+1)|)z^{t+1}$$
$$= \frac{1}{z}(g(z) - (n_{(1)}(0)|n_{(2)}(0)| \wedge |n_{(G)}(0)|))$$

where $(n_{(1)}(0)|n_{(2)}(0)| \wedge |n_{(G)}(0)|)$ is the initial manpower structure. Thus,

$$g(z)Qz = g(z) - (n_{(1)}(0)|n_{(2)}(0)| \wedge |n_{(G)}(0)|).$$

so that

$$g(z) = (n_{(1)}(0)|n_{(2)}(0)| \wedge |n_{(G)}(0)|)[I - Qz]^{-1} \quad (4)$$

Let

$$G(z) = [I - Qz]^{-1} = \sum_{t-0}^{\infty} Q^t z^t, Q^0 = I \quad (5)$$

where $G(z)$ is the $(\sum_{g-1}^{G} k_{(g)} \times \sum_{g-1}^{G} k_{(g)})$ Green function matrix and I is the $(\sum_{g-1}^{G} k_{(g)} \times \sum_{g-1}^{G} k_{(g)})$ identity matrix.
Define $k = \sum_{g-1}^{G} k_{(g)}$. Following the well-known Cayley-Hamilton theorem, Q is a root of the following characteristic polynomial in λ:

$$|\lambda I - Q| = \lambda^k - S_1\lambda^{k-1} + S_2\lambda^{k-2} + \wedge + (-1)^k S_K \quad (6)$$

where S_i is the sum of the principal minors of order k. Since $|\lambda I - Q| = \lambda^k |I - Q\frac{1}{\lambda}|$, then

$$|I - Q\frac{1}{\lambda}| = 1 - S_1\lambda^{-1} + S_2\lambda^{-2} + \wedge + (-1)^k S_K \lambda^{-k} \quad (7)$$

Define z to be $z = \lambda^{-k}$. So

$$|I - Qz| = 1 - S_1 z + S_2 z^2 + \wedge + (-1)^k S_K z^k \quad (8)$$

By the fundamental theorem of algebra, this polynomial in z can be factorised as

$$|I - Qz| = |Q|(\alpha_1 - z)(\alpha_2 - z) \wedge (\alpha_k - z) \quad (9)$$

Since Q is a stochastic matrix, then $z = 1$ is a root of the characteristic polynomial. Without a loss of generality, take $\alpha_1 = 1$. By the method of partial fractions, the inverse of $I - Qz$ can be written as

$$\frac{adj(I-Qz)}{|Q|(\alpha_1 - z)(\alpha_2 - z)\wedge(\alpha_k - z)} = \frac{C_1}{1-z} + \sum_{m-2}^{k} \frac{C_m}{\alpha_m - z} \tag{10}$$

where the C_m's are constant matrices, $m = 1, 2, 3, \wedge, k$. There may be instances where the α_m's are repeated. In such instances, some modifications are made to the above expression in line with the algebra of partial fractions. Using the series expansions, $\frac{1}{1-z} = \sum_{t-0}^{\infty} z^t$ and $\frac{1}{\alpha_m - z} = \sum_{t-0}^{\infty} \alpha_m^{-(1+t)} z^t$, we have

$$[I-Qz]^{-1} = C_1 \sum_{t-0}^{\infty} z^t + \sum_{m-2}^{k} (C_m \sum_{t-0}^{\infty} \alpha_m^{(1+t)} z^t) \tag{11}$$

Thus

$$[I-Qz]^{-1} = \sum_{t-0}^{\infty} z^t (C_1 \sum_{m-2}^{k} \alpha_m^{(1+t)} C_m) z^t \tag{12}$$

By comparing equation (5) and 12), we obtained

$$Q^t = C_1 + \sum_{m-2}^{k} \alpha_m^{(1+t)} C_m \tag{13}$$

Since Q is irreducible, then 1 is a simple root. It follows that

$$\varinjlim Q^t = C_1 + \varinjlim (\sum_{m-2}^{k} \alpha_m^{(1+t)} C_m) \tag{14}$$

and this implies that $\varinjlim(\sum_{m-2}^{k} \alpha_m^{(1+t)} C_m) = 0$ and C_1 is the steady-state matrix.
Proposition 2: As Q is stochastic, the fixed size manpower system is finite and bounded.

Proof
Let $(n_{(1)}(0)|n_{(2)}(0)| \wedge |n_{(G)}(0)|)$ be the base structure from which projections are to be made.
Consider the limiting structure of the system:

$$(n_{(1)}(\infty)|n_{(2)}(\infty)| \wedge |n_{(G)}(\infty)|) = (n_{(1)}(0)|n_{(2)}(0)| \wedge |n_{(G)}(0)|) \varinjlim Q^t \tag{15}$$

Recall that for any real matrix $A = (a_{ij})$ and a real vector $U = (u_j)$, their norms defined as $\|A\| = sup \sum_{j-1}^{k} a_{ij}$ and $\|U\| = sup|u_j|$ satisfies $\|UA\| < \|U\|\|A\|$

and $\|A\| = 1$ for any stochastic matrix A (cf. Dimitriou et al., 201 5). Since the transition matrix Q is stochastic, then

$$0 \leq \|(n_{(1)}(\infty)|n_{(2)}(\infty)| \wedge |n_{(G)}(\infty)|)\| \leq \|(n_{(1)}(0)|n_{(2)}(0)| \wedge |n_{(G)}(0)|)\| \| \varinjlim Q^t\|$$
$$= \|(n_{(1)}(0)|n_{(2)}(0)| \wedge |n_{(G)}(0)|)\| \varinjlim \|Q\|^t = \|(n_{(1)}(0)|n_{(2)}(0)| \wedge |n_{(G)}(0)|)\|$$

Thus,

$$0 \leq \|(n_{(1)}(\infty)|n_{(2)}(\infty)| \wedge |n_{(G)}(\infty)|)\| \leq \|(n_{(1)}(0)|n_{(2)}(0)| \wedge |n_{(G)}(0)|)\| \quad (16)$$

Which implies that the limiting structure $(n_{(1)}(\infty)|n_{(2)}(\infty)| \wedge |n_{(G)}(\infty)|)$ is bounded. The proof that $(n_{(1)}(\infty)|n_{(2)}(\infty)| \wedge |n_{(G)}(\infty)|)$ is finite is trivial as a fixed sized manpower system is modelled.
From the foregoing, let $\sigma N(t+1) = N(t+1) - N(0)$ where $N(t+1) = \sum_{g-1}^{G} \sum_{i-1}^{k_g} n_i(t+1)$ is the total size of the system at time $t+1$. By the proof of Proposition 2, the quantity $\varinjlim \sigma N(t+1)$ is not greater than zero in actual application.

3.2 *Numerical illustration*

The numerical application in this paper is a case-oriented study where data are collated from the archives of a typical university in Nigeria. For the Nigerian university system $G = 7$ for the academic staff as there are seven levels in the Consolidated University Academic Staff Salary (CONUASS) structure and the number of steps in a grade g are: $k_1 = 6; k_2 = 8; k_3 = 8; k_4 = 9; k_5 = 13; k_6 = 10; k_7 = 10$. The symbol g defines the ranks as follows: $g = 1$ denotes Graduate Assistant (GA), $g = 2$ denotes Assistant Lecturer (AL), $g = 3$ stands for Lecturer II (LII), $g = 4$ stands for Lecturer I (LI), $g = 5$ represents Senior Lecturer (SL), $g = 6$ represents Associate Professor (AP) and $g = 7$ denotes Professor (Prof).
We studied the academic manpower system at Igbinedion University, Okada. In this University, vacancies in any college are filled by either promotion from among serving employees of the University or by appointment in accordance with the regulations governing the service of senior staff in the University. For illustrative purposes, data on the academic staff stocks and flows were obtained for the following colleges: College of Natural and Applied Sciences, Basic Medical Sciences, Clinical Medicine, School of Pharmacy and College of Engineering. The data were aggregated according to the intra- and intergrade-specific feature of the system. We defined the data format to include the following subheadings: year, grade step, promotion and wastage. The data were identified from the University Bursary Department and collated as presented in Tables 1 – 4. For ease of identification, we have used superscripts to describe the kind of flows to and away from a grade. For instance, the superscript R was used to denote recruits, (h/j) was used to denote promotion to grade h step j and (0j) was used to represent the turnover (wastage) from step j in the current grade. Where no record was found for a particular feature, a dash is placed. In the data collection, it was assumed that decisions on leaving were made at the beginning of the academic year and that staff considered for promotion were selected from among those who have not indicated their intention to leave.

The tables show that: there were no demotions, recruits can come into the system at any grade level and step, there was an increase in step from one year to another, except when the staff has attained the highest step in that grade level, and has not obtained the requirements to move to the next higher grade level and there was no retrenchment (negative recruitment).

Table 1: Grade step sizes, transitions and turnover for Graduate Assistant (GA) over the years

Year	01/01	01/02	01/03	01/04	01/05	01/06	GA to LII	Wastage
2015	-	2	-	1	-	3	-	-
2016	-	1^R	1	-	1	3	$1^(3/2)$	-
2017	-	-	1	1	-	2	$1^{(3/2)} + 1^{(3/3)}$	-
2018	-	1^R	-	1	1	1	1(3/3)	-
2019	-	-	-	-	1	2	-	-
2020	-	-	-	-	-	2	1(3/1)	-

Table 2: Grade step sizes, transitions and turnover for Assistant Lecturer (AL) over the years

Year	02/01	02/02	02/03	02/04	02/05	02/06	02/07	02/08	AL to LII	Wastage
2015	-	4	1	4	2	1	-	2	-	-
2016	-	-	4	1	$4 + 1^R$	2	1	2	-	-
2017	-	1^R	1^R	4	1	4	2	3	$1^{(3/3)}$	-
2018	-	-	1	1	4	-	3	4	$1^{(3/4)}$	$1^{(05)+1(06)}$
2019	-	4^R	-	1	-	1	-	4	$3^{(3/2)} + 1^{(3/5)} + 2^{(3/6)}$	$1^{(04)}$
2020	1^R	1^R	4	1^R	-	-	1	4	-	$1^{(04)}$

Table 3: Grade step sizes, transitions and turnover for Lecturer II (LII) over the years

Year	03/01	03/02	03/03	03/04	03/05	03/06	03/07	03/08	LII to LI	Wastage
2015	-	7	3	6	6	4	-	3	-	-
2016	-	2^R	6	3	6	6	4	3	-	$1^{(02)}$
2017	-	3^R	$2 + 4^R$	6	2	5	4	7	$1^{(4/2)}$	$1^{(04)} + 2^{(06)}$
2018	-	3^R	$3 + 1^R$	$6 + 2^R$	6	2	4	5	$2^{(4/3)} + 3(4/2)$	$1^{(07)} + 1^{(08)}$
2019	-	3^R	$3 + 1^R$	$4 + 1^R$	7	2	-	6	$1^{(4/1)} + 4^{(4/2)} + 2^{(4/3)}$	$1^{(05)} + 1^{(06)} + 1^{(04)}$
2020	1^R	-	$2 + 6^R$	4	5	$6 + 1^R$	-	6	$1^{(4/2)}$	$1^{(02)} + 1^{(05)} + 1^{(06)}$

Table 4: Grade step sizes, transitions and turnover for Lecturer I (LI) over the years

Year	04/01	04/02	04/03	04/04	04/05	04/06	04/07	04/08	04/09	LI to SL	Wastage
2015	-	11	7	5	4	1	4	3	3	-	-
2016	-	-	10	7	$5 + 1^R$	4	1	$3 + 1^R$	6	-	$1^{(02)} + 1^{(07)}$
2017	-	7^R	-	7	5	6	3	1	$8 + 1^R$	$1^{(5/2)}$	$3(03) + 1^{(04)} + 1^{(06)+2(09)}$
2018	-	5^R	6	1^R	6	3	4	2	8	$3^{(5/2)} + 1^{(5/3)}$	$1^{(02)} + 1^{(04)} + 1^{(06)} + 2^{(09)}$
2019	-	1^R	$5 + 3^R$	$6 + 1^R$	1	4	2	4	6	$3^{(5/2)}$	$2^{(05)} + 1^{(08)} + 1^{(09)}$
2020	2^R	2^R	$1 + 6^R$	8	5	1	4	2	9	-	$2^{(04)} + 1^{(09)}$

For (full) professor, we omitted the intra-mobility hereinafter as this rank is the terminal position in the academic staff structure. Thus, there was no need to unbundle the grade level so that $k_{(7)} = 1$. By the usual assumption that the flows follow a multinomial distribution (Bartholomew et al., 1991), the various transition probabilities were estimated based on the method of maximum likelihood from the historical data set. We obtained and compiled the results of the different transitions in the system in a matrix form. The flow matrices were very large and sparse. For the sake of brevity, our computations and compilations were presented using the MATLAB pseudo code.

In the pseudo code, the transition matrices $\Phi^{(g)}_{k_{(g)}\times k_{(g)}}, g = 1, 2, \wedge, 7$, were represented by GA, AL, LII, LI, SL and AP and they showed the flow pressures concerning the intra-grade (horizontal) mobility in the system, and the matrices $\Phi^{(g+h)}_{k_{(g)}\times k_{(g+h)}}, g, h = 1, 2, \wedge, 7, g \neq h$, were denoted by GLII, ALII, LLI, LSL, SAP and AProf. The horizontal mobility matrices are sub-stochastic matrices, while the vertical mobility matrices are neither stochastic nor sub-stochastic matrices. The transition matrix P, which is a block matrix containing the horizontal and vertical mobility matrices, provided information on the internal workforce flows in the university system. This matrix is sub-stochastic. The sub-stochastic characteristic nature of the matrices was due to the turnovers (such as retirement, resignation, termination of appointment, etc.) in the system. According to the assumption that recruitment is done to replace wastage, the Markov chain that governs the entire system is the 55 by 55 transition matrix Q. This matrix Q is stochastic. Moreover, 1 is a simple eigenvalue of Q and so the steady-state distribution of the matrix Q exists, which makes it suitable for use for a long-term projection of the structure of the system.

Without a loss of generality, we let $t = 0$ denote the starting time point of the extrapolation of the manpower structure. In this illustration, this was the base year 2020, where $N(0) = 92$. We also extrapolate the structure of the system for $t = 1, 2, \wedge$. We numerically computed the expected structure of the system for the next five years. The results were collated in Tables 5 – 9. The results showed that the numbers in the grades changes relatively slightly from that of the base year, except for LI and Prof. The system can therefore be referred to as a transient system. Observed that $\sigma N(t+1) < 0$, which implies that the number of wastage tends to be higher than the recruitments. Thus, there is a need to outsource personnel in order to attain the fixed total size of 92 academic staff for the colleges. Personnel may be outsourced by way of sabbatical, adjunct, part-time or on contract basis. For $t = 1, 2, 3$, one additional academic staff should be outsourced; while for $t = 4, 5$, four additional academic staff are required. From the evolution of the system through time, the administration of the system may be able to determine whether the existing policy would result to a bottom-heavy or a top-heavy structure in accordance with the statutory staff-mix by rank of the National Universities Commission (NUC) cf. Ekhosuehi and Omosigho (2018). From the extrapolated structure, the system follows a pyramidal structure as $\sum_{g-1}^{4} N_g(t) \geq N_5(t) \geq \sum_{g-6}^{7} N_g t$. Since there was a disproportionately large number of staff in the lower grades, the system will be bottom-heavy, if the current policy of the system persists.

Table 5: Expected grade step sizes and structure for $t = 1$

Grade	01	02	03	04	05	06	07	08	09	10	11	12	13	$N_g(1)$
GA	0	0	0	0	0	1								1
AL	0	1	0	4	1	0	0	4						10
LII	0	1	1	7	4	5	4	4						26
LI	0	4	2	1	7	7	1	4	9					35
SL	0	2	1	0	2	3	1	1	1	0	0	0	3	14
AP	0	0	0	0	0	0	0	0	0	2				2
Prof														3
													Total	91

Table 6: Expected grade step sizes and structure for $t = 2$

Grade	01	02	03	04	05	06	07	08	09	10	11	12	13	$N_g(2)$
GA	0	0	0	0	0	1								1
AL	0	1	1	0	3	1	0	3						9
LII	0	1	2	2	7	4	3	6						25
LI	0	3	5	2	1	6	5	1	11					34
SL	0	2	3	1	0	2	2	1	1	1	0	0	3	16
AP	0	0	0	0	0	0	0	0	0	2				2
Prof														4
													Total	91

Table 7: Expected grade step sizes and structure for $t = 3$

Grade	01	02	03	04	05	06	07	08	09	10	11	12	13	$N_g(3)$
GA	0	0	0	0	0	1								1
AL	0	1	1	1	0	2	1	2						8
LII	0	2	2	3	2	6	3	6						24
LI	0	4	5	5	2	1	5	4	9					35
SL	0	2	3	2	1	0	2	1	1	1	1	0	3	17
AP	0	0	0	0	0	0	0	0	0	2				2
Prof														4
													Total	91

Table 8: Expected grade step sizes and structure for $t = 4$

Grade	01	02	03	04	05	06	07	08	09	10	11	12	13	$N_g(4)$
GA	0	0	0	0	0	1								1
AL	0	1	1	1	1	0	2	2						8
LII	0	1	2	2	3	2	4	6						20
LI	0	3	6	5	5	2	1	4	11					37
SL	0	1	3	2	1	1	0	1	1	1	1	1	3	16
AP	0	0	0	0	0	0	0	0	0	2				2
Prof														4
													Total	88

Table 9: Expected grade step sizes and structure for $t = 5$

Grade	01	02	03	04	05	06	07	08	09	10	11	12	13	$N_g(5)$
GA	0	0	0	0	0	1								1
AL	0	1	1	1	1	1	0	3						8
LII	0	2	2	2	2	3	1	7						19
LI	0	3	5	6	5	4	2	1	12					38
SL	0	2	2	2	1	1	1	0	1	1	1	1	3	16
AP	0	0	0	0	0	0	0	0	0	2				2
Prof														4
													Total	88

Table 10: Summary of Extrapolated Expected Structure of the System

Staff categories	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
Lecturer 1 and below	79.1%	75.8%	74.7%	75%	75%
Senior lecturer	15.4%	17.6%	18.7%	18.2%	18.2%
Professorial cadre	5.5%	6.6%	6.6%	6.8%	6.8%

The results in table 10 violate completely the staff mix by rank of 45:35:20 as stipulated by National Universities Commission (NUC). It is obvious from the results that the distribution has not attained the steady state required to obtain the stipulated guideline. The system is bottom heavy, therefore lacks the necessary personnel experience for now. This is a great problem during NUC Accreditation visit to the Institution. As a remedy to the above, recruitment and outsourcing of staff should be made into the senior lecturer and Professorial cadres.

4. Conclusion

This work used Markov chain as a tool to describe the evolution of a hierarchical manpower system. The notion of intra/inter-grade mobility in a fixed size manpower system has been investigated. In such a system, the manpower structure was finite and bounded as the transition matrix of the system was stochastic and irreducible. An illustrative example was sourced using practical data from a private university in Nigeria to illustrate the new method. The real life example demonstrates that the technique of attributing the same promotion probability to each member of the same grade can be advanced further to the grade step. It is hoped that this method can be used in any multi-step G-grade with intra/inter-grade manpower system in making prudent decisions regarding the human resource needs of the organization.

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