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Transmuted Rayleigh-X Family of Distributions

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Abstract. Real world phenomena are always described and predicted using distributions, due to these usefulness distributions are developed and studied by different researchers. In this research, we introduced a novel class of Statistical distributions called Transmuted Rayleigh-X (TR-X). The study focused on exploring various statistical properties of the proposed class, including moment, moment generating function, limiting behaviour, quantile function, distribution of order statistics, reliability and order relevant characteristics. For better understanding of the TR-X class of distributions, we generated plots for the cumulative distribution function (CDF), probability density function (PDF), survival and hazard function for a sub-model (Transmuted Rayleigh-Exponential Distribution) within the family. These plots were constructed for different parameters values to visualize how the distribution changes with varying inputs. The sub-model from the family provided valuable insights for potential applications in diverse fields where probability distributions play a crucial role. The performance of the sub-model was tested using two different datasets. The sub model when compared with Exponential and its extensions revealed a better fit for the two datasets. Therefore, this research concludes that the TR-X family of distributions is capable of generating flexible and robust distribution that can be fitted to data of different shapes.

Keywords: Transform-transformer method, entropy, order statistics, hazard function

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1. Introduction

Statistical distributions have found widespread applications across various fields of knowledge. They are used to fit datasets, enabling researchers and practitioners to draw valid conclusions from the data. By applying appropriate statistical distributions, professionals can model and analyse data efficiently, make predictions, and make valid decisions in fields such as economics, finance, engi-

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neering, social sciences, biology, medicine,and many others. The flexibility and utility of statistical distributions make them an indispensable tool in data analysis and research. Every day problems arises in these fields as the data generated cannot be fitted to the existing distributions. As such researchers continue to developed new methods to generate distributions and families of distributions. (Reyad and Othman 2017, Modi and Gill, 2019 and Usman et al 2019).

The work of Tahir et al. (2015) also indicated that extended distributions have flexibility in their properties, as such both theoretical and statisticians developed interest in proposing generalized families of distributions. To enhance the flexibility and accuracy of the existing distributions, one or more parameter(s) are incorporated into the existing model. This allows the distribution to better adapt and fit to the data, resulting in improved performance and a more robust representation of the underlying data patterns. A number of these distributions and families of distributions are generated by different researchers using different methods, some of which are: Eugene et al. (2002) developed a method of generating new family of distributions and called it beta-generated method. The CDF has two extra shape parameters. In the literature, researchers have investigated various families and distributions using the beta distribution as a basis. Some notable example of such distributions and families include; the beta- generalized logistic by Morais et al.(2013), the beta compound applied Rayleigh distribution by Reyad and Othman (2017), the beta exponential frechet distribution by Mead et al. (2017), the Beta Weibull-G family of distribution by Yousof et al. (2017) and many others.

In 2011, Cordeiro and de Castro introduced a novel extension to the beta-generated method, which they named the kumaraswamy-generated method. This new approach involved replacing the beta random variable used in the original method with that of the Kumaraswamy distribution. Various families and compound distributions have been formulated using the Kumaraswamy distribution as a building block. Some notable example are as follow; the Kumaraswamy Gompertz Makeham distribution proposed by Chukwu and Ogunde in 2016, and Kumaraswamy log-logistic Weibull distribution examined by Mdlongwa et al. in (2019).

Shaw and Buckley (2007) introduced a method for creating novel families of distributions called the Transmuted family of distributions. This approach has garnered significant attention from researchers, leading to the development of several families and distributions using this technique. Some notable Examples include The transmuted exponentiated generalized-G family of distributions derived by Yousof et al. (2015), the Transmuted Gompertz distribution presented by AbdulMoniem and Seham (2015) and Transmuted Kumaraswamy distribution proposed by Shuaib et al (2016).

Alzaatreh et al. (2013), introduced a general approach for developing new classes of probability distributions as well as new distributions and named it the Transformed- Transformer Method, for short the T-X method. Since the introduction of this technique, many researchers applied it among them are; Alzaghal et al. (2013) provided an extension of the T-X method and named it as the exponentiated T-X family of distributions, another researchers named Aljarrah et al. (2014) developed the generalised T-X family of distribution by considering the quantile function, Alzaatreh et al. (2014), generalized the

normal distribution to a family called T-normal family of distributions, Mohammed et al. (2020) studied the properties of of generalized odd generalized exponential-exponential distribution, Mohammed and Ugwuowo (2020) introduced the Transmuted Exponential-G family of distribution, Mohammed et al. (2022) studied Generalized Transmuted Exponential-Exponential Distribution and its applications, Yahaya and Doguwa (2021) proposed Rayleigh- Exponentiated odd generalized-X family of distribution, Mohammed and Ugwuowo (2021) Transmuted Exponential-Topp Leon distribution with monotonic and non-monotonic hazard rate.
The article discusses a novel class of model by utilizing the idea of Alzaatreh et al. (2013) using Transmuted Rayleigh Distributions as the Transform (T) random variable. The new family is termed as Transmuted Rayleigh-X (TR-X) family of distributions, where X serve as the Transform random variable. It is characterised by two extra parameters which are considered as scale and transmuted parameters. These parameters increased the flexibility of the newly model generated within this family. Through out the article, various structural properties of the new generated family are derived and explained.

2. Materials and Method

2.1 *The Transmuted Rayleigh-X Class of Distributions*

Consider a random variable, that follows the Transmuted Rayleigh distribution (Merovci, 2013). This distribution is characterised by its distribution and density function respectively presented as;

$$R(t) = \left(1 - e^{-\frac{t^2}{2\sigma^2}}\right)\left(1 + \lambda e^{-\frac{t^2}{2\sigma^2}}\right) \quad and \quad r(t) = \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}} \left(1 - \lambda + 2\lambda e^{-\frac{t^2}{2\sigma^2}}\right),$$

with $\sigma, \lambda, t > 0$.
Let now consider any baseline distribution with a random variable denoted by , with CDF and PDF respectively given by $G(x;\varphi)$ and $g(x;\varphi)$ indexed by say, q-parameter vector $\varphi = (\varphi_1, \varphi_2, ...\varphi_q)$. Using a link function given as $WG(x;\varphi) = -\log(1 - G(x;\varphi))$ from Alzaatreh et al. (2013). Then the distribution function of the novel generated class of distribution is defined as;

$$F(x, \sigma, \lambda, \varphi) = \int_0^{WG(x;\varphi)} r(t)\, dt \tag{1}$$

By substituting the value of $WG(x;\varphi)$ and $r(t)$ in equation (1) and integrating with respect to t we obtained the CDF as;

$$F(x, \sigma, \lambda, \varphi) = \left(1 - e^{-\frac{(-\log(1-G(x;\varphi)))^2}{2\sigma^2}}\right)\left(1 + \lambda e^{-\frac{(-\log(1-G(x;\varphi)))^2}{2\sigma^2}}\right) \tag{2}$$

The PDF for the TR-X class of distributions is derived from equation (2) as follows;

$$f\left(x,\sigma,\lambda,\varphi\right)=\frac{g\left(x;\varphi\right)\log\left(1-G(x;\varphi)\right)}{\sigma^{2}\left(1-G(x;\varphi)\right)}e^{-\frac{1}{2\sigma^{2}}\left(-\log(1-G(x;\varphi))\right)^{2}}\left[1-\lambda+2\lambda e^{-\frac{1}{2\sigma^{2}}\left(-\log(1-G(x;\varphi))\right)^{2}}\right] \tag{3}$$

3. Linear Representation

In this sub-section, the article presents straightforward expressions for the CDF and PDF of the TR-X family of distributions. These expressions are derived using the concepts of generalized binomial and power series expansions. These mathematical techniques are employed to facilitate the determination of various statistical properties associated with this newly family of distributions.
Let consider the CDF (2) of TR-X family of distributions expressed as;

$$F(x;\sigma,\lambda,\varphi)=1+\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}b_{i,k}G(x;\varphi)^{\beta}=1+\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}b_{i,k}H_{\beta}(x;\varphi) \tag{4}$$

where $b_{i,k}=\left(\left(\frac{1}{2\sigma^{2}}\right)^{i}(\lambda-1)-\left(\frac{1}{\sigma^{2}}\right)^{i}(\lambda)\right)2i\binom{k-2i}{k}\sum_{j=0}^{k}\frac{(-1)^{i+j+k}\binom{k}{j}p_{j,k}}{(2i-j)}$ and $G(x;\varphi)^{\beta}=H_{\beta}(x;\varphi)$ represent the distribution function of the Exponentiated-G (Exp-G) class of distribution associated with $\beta=(2i+k)$ as the power parameter. The PDF associated with equation (8) is obtained as;

$$f\left(x,\sigma,\lambda,\varphi\right)=\sum_{i,k=0}^{\infty}b_{i,k}\beta g(x;\varphi)G(x;\varphi)^{\beta-1}=\sum_{i,k=0}^{\infty}b_{i,k}h_{\beta}\left(x;\varphi\right) \tag{5}$$

Where $h_{\beta}(x;\varphi)=\beta g(x;\varphi)G(x;\varphi)^{\beta-1}$ represent the density function of the Exp-G class of distributions with $\beta=(2i+k)$ as the power parameter.

4. Statistical Properties of TR-X Family of Distributions

In this sub-section, we examined various statistical characteristics the TR-X class of distributions.These properties include the moments, moment generating function (MGF), limiting behaviour, quantile function (QF), distribution of order statistics, reliability and more.

4.1 *The Moments*

The TR-X family of distributions moments refer to as r^{th} the ordinary moments denoted by $\mu_{r}^{'}$ can be define as follows;

$$\mu_r^{'} = E(X^r) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k} E(Z_{i,k}^r) \tag{6}$$

for, r = 1,2, ...
where $E\left(Z_{i,k}^r\right) = \int_0^\infty x^r h_\beta\left(x;\varphi\right) dx$ and $Z_{i,k}$ represents the Exp-G class of distribution with $2i + k$ as its parameter.

4.2 *The Incomplete Moments*

The TR-X family of distributions incomplete moment denoted by $I_r(y)$ is define as follows;

$$I_r(y) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k} I_{i,k}(y) \tag{7}$$

where $I_{i,k}(y) = \int\limits_0^y x^r h_\beta\left(x;\varphi\right) dx$

4.3 *The Moment Generating Function (MGF)*

The MGF of the TR-X family of distribution is define by the following equation;

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k} E\left(e^{tZ_{i,k}}\right) \tag{8}$$

where, $E(e^{tZ_{i,k}}) = \int\limits_0^\infty e^{tX_{i,k}} h_\beta\left(x;\varphi\right) dx$

4.4 *The Quantile Function (QF)*

The QF of the TR-X family of distributions is expressed as follows;

$$Q_x\left(u\right) = G^{-1}\left(1 - e^{-\left(2\sigma^2\left(-\log\left(\frac{(\lambda-1)\pm\sqrt{(\lambda-1)^2+(4\lambda(1-u))}}{2\lambda}\right)\right)\right)^{1/2}}\right) \tag{9}$$

where $G^{-1}\left(.\right)$ represent the QF of the random variable *X* .
Now , equation (9) is the quantile function of the TR-X family of distributions. Setting specific values say u =0.25 , u = 0.5 and u = 0.75 in this equation allows us to obtain the first quartile, the median and third quartile of the TR-X family

of distribution, respectively.

$$Q_x(0.25) = G^{-1}\left(1 - e^{-\left(2\sigma^2\left(-\log\left(\frac{(\lambda-1)\pm\sqrt{\lambda^2+\lambda+1}}{2\lambda}\right)\right)\right)^{1/2}}\right) \tag{10}$$

$$Q_x(0.5) = G^{-1}\left(1 - e^{-\left(2\sigma^2\left(-\log\left(\frac{(\lambda-1)\pm\sqrt{\lambda^2+1}}{2\lambda}\right)\right)\right)^{1/2}}\right) \tag{11}$$

$$Q_x(0.75) = G^{-1}\left(1 - e^{-\left(2\sigma^2\left(-\log\left(\frac{(\lambda-1)\pm\sqrt{\lambda^2-\lambda+1}}{2\lambda}\right)\right)\right)^{1/2}}\right) \tag{12}$$

4.5 *The Entropy*

In this situation, we will explore the Renyi entropy, which serves as a measure of variation or uncertainty of a random variable. Renyi entropy finds application in engineering, sciences and probability theory. (Renyi, 1961).
For a random variable X that follows the TR-X family of distribution, the Renyi entropy and q-entropy are respectively expressed as:

$$I_\theta(x) = \frac{1}{1-\theta}\log\left\{\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k}\right)^{\theta}\int_0^\infty \left(h_\beta(x;\varphi)\right)^{\theta} dx\right\}; \quad \theta > 0 \ and \ \theta \neq 1 \tag{13}$$

and

$$H_q(x) = \frac{1}{q-1}\log\left\{1 - \left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k}\right)^{q}\int_0^\infty \left(h_\beta(x;\varphi)\right)^{q} dx\right\}; \quad q \neq 1 \ and \ q > 0 \tag{14}$$

4.6 *Distribution of Order Statistics*

Suppose we have a random sample say $X_1, ..., X_n$ from the TR-X family of distributions and let be the corresponding order statistics obtained from this sample. In that case, we can express the PDF of the p^{th} order statistic, $X_{p:n}$ as follows:

$$f_{p:n}(x) = \sum_{l=0}^{n-p}\frac{(-1)^l n!}{(p-1)!\,(n-p-l)!l!}\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k}h_\beta(x;\varphi)\right)\left(1 + \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k}H_\beta(x;\varphi)\right)^{p+l-1} \tag{15}$$

The formulas to calculate the X_1 and X_n are as follows:

$$f_{1:n}(x) = \frac{n!}{(n-1)!}\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k}h_\beta(x;\varphi)\right)\left(-\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k}H_\beta(x;\varphi)\right)^{n} \tag{16}$$

and

$$f_{n:n} = \frac{n!}{(n-1)!}\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k} h_{\beta}(x;\varphi)\right)\left(1+\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k} H_{\beta}(x;\varphi)\right)^{n-1} \tag{17}$$

4.7 *The Complimentary CDF (Survival Function)*

$$S(t) = e^{-\frac{1}{2\sigma^2}(-\log(1-G(t;\varphi)))^2}\left(1-\lambda+\lambda e^{-\frac{1}{2\sigma^2}(-\log(1-G(t;\varphi)))^2}\right) \tag{18}$$

4.8 *The Failure Rate (Hazard Function)*

The *h(t)* for the proposed family of distribution is calculated by using the formula provided below ;

$$h(t) = \frac{\frac{g(t;\phi)\log(1-G(t;\phi))}{\sigma^2(1-G(t;\phi))} e^{-\frac{(-\log(1-G(t;\phi)))^2}{2\sigma^2}}\left[1-\lambda+2\lambda e^{-\frac{(-\log(1-G(t;\phi)))^2}{2\sigma^2}}\right]}{e^{-\frac{(-\log(1-G(t;\phi)))^2}{2\sigma^2}}\left[1-\lambda+\lambda e^{-\frac{(-\log(1-G(t;\phi)))^2}{2\sigma^2}}\right]} \tag{19}$$

5. Estimation of Parameters for TR-X class of Distributions

Here we are using the method of maximum likelihood estimation (MLE) to estimate the unknown parameters of the proposed family of distributions known as the TR-X. Let assumed we have a samples random variables $X_1, X_2, ..., X_n$ from a population that follows TR-X family of distributions, having the joint PDF of these samples represented $f(x_1, x_2, ..., x_n; \Theta)$ as $\Theta = (\sigma, \lambda, \varphi)^T$ is a vector of unknown parameter that we want to estimate.
Then the likelihood function denoted by $L(\Theta)$ for these samples is expressed as:

$$L(\Theta) = \prod_{i=1}^{n}\left\{\frac{g(x;\varphi)\log(1-G(x;\varphi))}{\sigma^2(1-G(x;\varphi))} e^{-\frac{1}{2\sigma^2}(-\log(1-G(x;\varphi)))^2}\left[1-\lambda+2\lambda e^{-\frac{1}{2\sigma^2}(-\log(1-G(x;\varphi)))^2}\right]\right\} \tag{20}$$

Taking the natural log of $L(\Theta)$, gives the log-likelihood denoted by ℓ

$$\begin{aligned}\ell = &\sum_{i=1}^{n}\log g(x_i;\varphi) + \sum_{i=1}^{n}\log(1-G(x_i;\varphi)) - 2n\log\sigma - \sum_{i=1}^{n}\log(1-G(x_i;\varphi)) \\ &- \frac{1}{2\sigma^2}\sum_{i=1}^{n}(-\log(1-G(x_i;\varphi)))^2 + \sum_{i=1}^{n}\log\left(1-\lambda+2\lambda e^{-\frac{1}{2\sigma^2}(-\log(1-G(x_i;\varphi)))^2}\right)\end{aligned} \tag{21}$$

To obtain the component of the score vector denoted by $U(\Theta)$, we calculate it by taking partial derivatives of the with respect to the parameters λ, σ and φ . Then we set the resulting derivatives equal to zero.

$$U(\Theta) = \frac{\partial \ell}{\partial \Theta} = \left(\frac{\partial \ell}{\partial \varphi}, \frac{\partial \ell}{\partial \varphi}, \frac{\partial \ell}{\partial \varphi}\right)^T$$

$$U_\lambda = \frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \frac{2e^{-\frac{1}{2\sigma^2}(-\log(1-G(x_i;\varphi)))^2} - 1}{1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2}(-\log(1-G(x_i;\varphi)))^2}} \tag{22}$$

$$\begin{aligned} U_\sigma = \frac{\partial \ell}{\partial \sigma} &= \frac{-2n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}\left(-\log\left(1 - G\left(x_i;\varphi\right)\right)\right)^2 \\ &+ \frac{2\lambda e^{-\frac{1}{2\sigma^2}(-\log(1-G(x_i;\varphi)))^2}\sigma^{-3}\left(-\log\left(1 - G\left(x_i;\varphi\right)\right)\right)^2}{1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2}(-\log(1-G(x_i;\varphi)))^2}} \end{aligned} \tag{23}$$

$$\begin{aligned} U_\varphi = \frac{\partial \ell}{\partial \varphi} &= \sum_{i=1}^{n}\frac{g'(x_i;\varphi)}{g(x_i;\varphi)} + \sum_{i=1}^{n}\frac{g(x_i;\varphi)}{\overline{G}(x_i;\varphi)} - \sum_{i=1}^{n}\frac{g(x_i;\varphi)}{\left(1 - G\left(x_i;\varphi\right)\right)\left(\log\left(1 - G\left(x_i;\varphi\right)\right)\right)} \\ &- \sum_{i=1}^{n}\frac{1}{\sigma^2}\frac{\left(-\log\left(1 - G\left(x_i;\varphi\right)\right)\right)g(x_i;\varphi)}{\overline{G}(x_i;\varphi)} + \sum_{i=1}^{n}\frac{\frac{-g(x_i;\varphi)(-\log(1-G(x_i;\varphi)))}{\sigma^2\overline{G}(x_i;\varphi)}e^{-\frac{1}{2\sigma^2}(\log(1-G(x_i;\varphi)))^2}}{1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2}(\log(1-G(x_i;\varphi)))^2}} \end{aligned} \tag{24}$$

6. The Transmuted Rayleigh Exponential Distribution (TR-ED)

Here, a novel model (TR-ED) was developed from the TR-X Class of distributions. Where exponential distribution (ED) served as the parent model which is characterized with the CDF and PDF provided in (25) and (26) respectively.

$$G(x;\vartheta) = 1 - e^{-\vartheta x} \tag{25}$$

and

$$g(x;\vartheta) = \vartheta e^{-\vartheta x} \tag{26}$$

6.1 *The TR-ED CDF and PDF*

The CDF of the TR-ED is obtained as;

$$F(x,\sigma,\lambda,\vartheta) = \left(1 - e^{-\frac{\left(-\log\left(1-\left(1-e^{-\vartheta t}\right)\right)\right)^2}{2\sigma^2}}\right)\left(1 + \lambda e^{-\frac{\left(-\log\left(1-\left(1-e^{-\vartheta t}\right)\right)\right)^2}{2\sigma^2}}\right) \tag{27}$$

The corresponding PDF is obtained as;

$$\begin{aligned} f(x,\sigma,\lambda,\vartheta) = &\frac{\left(\vartheta e^{-\vartheta x}\right)\log\left(1 - \left(1 - e^{-\vartheta x}\right)\right)}{\sigma^2\left(1 - \left(1 - e^{-\vartheta x}\right)\right)}e^{-\frac{1}{2\sigma^2}\left(-\log\left(1-\left(1-e^{-\vartheta x}\right)\right)\right)^2} \\ &\times\left[1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2}\left(-\log\left(1-\left(1-e^{-\vartheta x}\right)\right)\right)^2}\right] \end{aligned} \tag{28}$$

Figures 1 and 2 displays the behaviour of the CDF and PDF of the TR-ED for several different values of the parameters, to maintained simplicity in tne presentation the parameters are replace with specific values as $\sigma = a$, $\vartheta = d$ and $\lambda = b$.

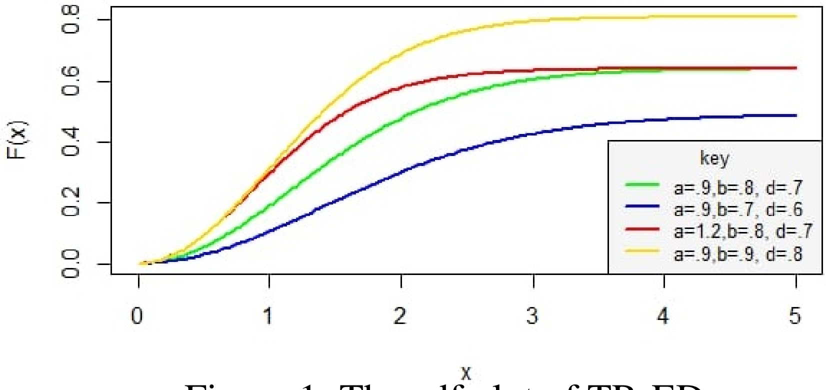

Figure 1: The cdf plot of TR-ED

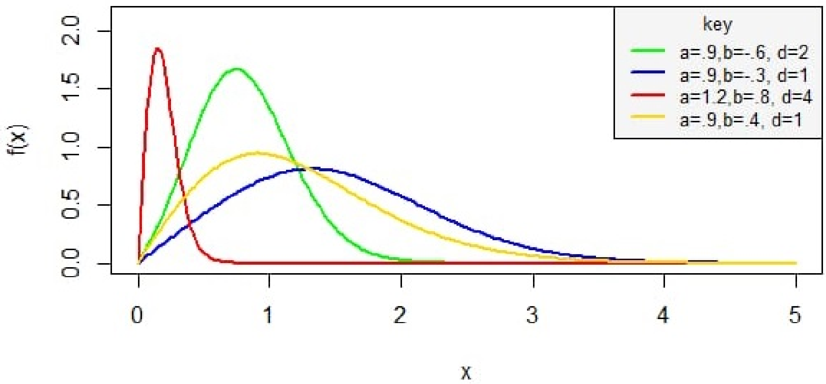

Figure 2: The pdf plot of TR-ED

6.2 *Survival function of TR-ED*

The survival function of the TR-ED is given as;

$$S\left(t;\sigma,\lambda,\vartheta\right) = 1 - \left(1 - e^{-\frac{\left(-\log\left(1-\left(1-e^{-\vartheta t}\right)\right)\right)^2}{2\sigma^2}}\right)\left(1 + \lambda e^{-\frac{\left(-\log\left(1-\left(1-e^{-\vartheta t}\right)\right)\right)^2}{2\sigma^2}}\right) \tag{29}$$

6.3 *Hazard function of TR-ED*

The hazard function is expressed as;

$$h\left(x;\sigma,\lambda,\vartheta\right) = \frac{-\left(\frac{\vartheta}{\sigma}\right)^2 x e^{-\frac{1}{2}\left(\frac{\vartheta x}{\sigma}\right)^2}\left[1 - \lambda + 2\lambda e^{-\frac{1}{2}\left(\frac{\vartheta x}{\sigma}\right)^2}\right]}{1 - \left(1 - e^{-\frac{1}{2}\left(\frac{\vartheta x}{\sigma}\right)^2}\right)\left(1 + \lambda e^{-\frac{1}{2}\left(\frac{\vartheta x}{\sigma}\right)^2}\right)} \tag{30}$$

The presentations of the behaviour of the complementary CDF and hazard functions of the TR-ED for some varying parameters values are given in Figures 3 and 4.

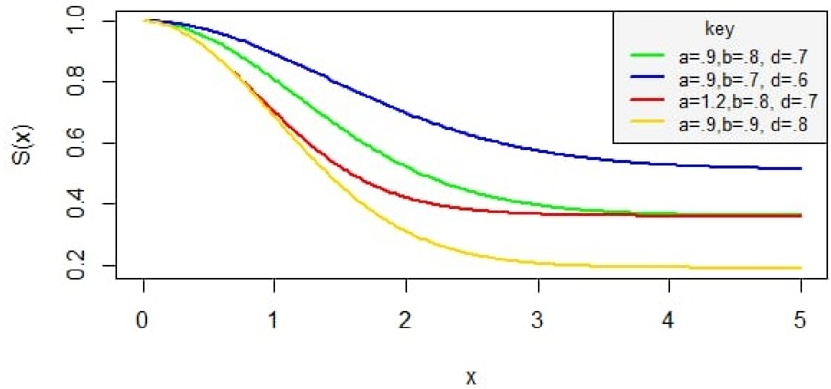

Figure 3: The survival function plot of TR-ED

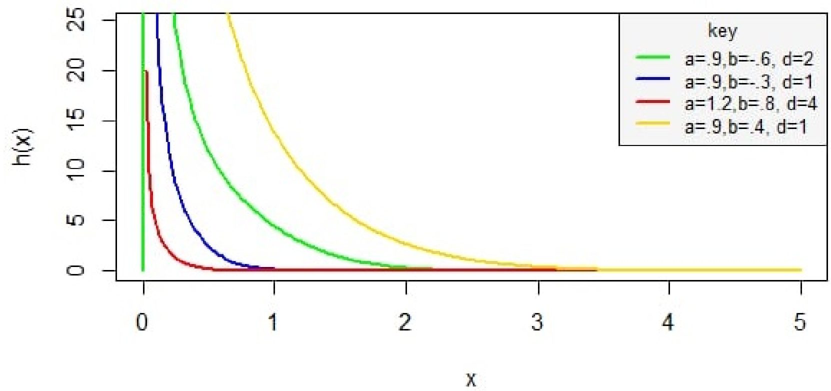

Figure 4: The hazard function plot of TR-ED

6.4 *Estimation of Parameters of the TR-ED*

Here, the new generated distribution parameters are computed using the MLE technique. With sample values $x_1, x_2, ..., x_n$ having joint density functions defined as $f\left(x_1, x_2, ..., x_n; \Theta\right)$ and the vector of an unknown parameter is denoted by $\Theta = (\sigma, \lambda, \vartheta)^T$. The ℓ for random variable $X_1, X_2, ..., X_n$ that follows the TR-ED can be described as follows:

$$\begin{aligned}\ell = &\sum_{i=1}^{n} \log\left(\vartheta e^{-\vartheta x}\right) + \sum_{i=1}^{n} \log\left(e^{-\vartheta x}\right) - 2n\log\sigma - \sum_{i=1}^{n} \log\left(e^{-\vartheta x}\right) \\ &- \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(-\log e^{-\vartheta x}\right)^2 + \sum_{i=1}^{n} \log\left(1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2}\left(-\log e^{-\vartheta x}\right)^2}\right)\end{aligned} \tag{31}$$

By performing partially differentiation for ℓ with regard to the parameters and equating the result to 0, one can derive the component of the score vector, .

7. Results and Discussion

7.1 *Applications*

In this study, the new distribution (TR-ED) was practicalized on two different real-life data to assess its flexibility. The parameters of the model are estimated using MLE through utilization of R software. The performance of the generated TR-ED, which belongs to the TR-X, was compared to that of the Transmuted Exponential Distribution (TE-ED), Beta Exponential Distribution (B-ED), Kumaraswamy Exponential Distribution (K-ED) and Exponential Distribution (ED) obtained from the existing literature using two real dataset.

7.2 *Statistics used for Comparing the Distributions*

In our comparison of the proposed model with its competitors, we considered several measures, including the Akaike Information Criterion (AIC), Anderson-Darling (A^*) and Cramer-Von Mises (W^*). The model that exhibits the lowest value of these statistics is considered the most favourable or best -fitting model among the models we evaluated.

7.3 *The Datasets*

The first dataset was on the survival time in years of some patient who survived after receiving chemotherapy . Bekker et al. (2000) also applied it. 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033

Table 1: Estimates of the TR-ED and the competing models parameters for the first Data

Model	$\widehat{\sigma}$	$\widehat{\lambda}$	$\widehat{\vartheta}$	$\widehat{\alpha}$	$\widehat{\beta}$
TR-ED	0.0004	0.9147	0.2345	-	-
TE-ED	-	-0.01455	0.6182	1.2143	-
B-ED	-	-	1.1045	1.02862	0.7726
K-ED	-	-	1.7608	0.1081	7.2869
ED	-	-	0.7455	-	-

Table 2: Present the outcomes of the Goodness-of-Fit Statistics for the first Dataset

Model	$-\ell$	AIC	W^*	A^*	$RANK$
TR-ED	-597.1238	-1188.2480	0.0579	0.4199	1
TE-ED	58.2181	122.4361	0.0793	0.5312	4
B-ED	58.0949	122.1899	0.0786	0.5270	3
K-ED	-57.6372	121.2745	0.0617	0.4257	2
ED	58.2186	118.4372	0.0788	0.5284	5

The second dataset utilized consisted of remission times of a random sample of 128 bladder cancer patients. Oguntunde et. al (2016) also used the data. 0.08, 2.09, 3.48,4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57,5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54,3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64,3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05,2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26,2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02,4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25,8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07,3.36, 6.93, 8.65, 12.63, 22.69.

Table 3: Estimates of the TR-ED and the competing models parameters for the second Data

Model	$\widehat{\sigma}$	$\widehat{\lambda}$	$\widehat{\vartheta}$	$\widehat{\alpha}$	$\widehat{\beta}$
TR-ED	0.0021	0.7738	4.6893	-	-
TE-ED	-	-0.2526	0.4904	0.2466	-
B-ED	-	-	1.3608	0.2713	0.4309
K-ED	-	-	1.4534	0.2882	0.4053
ED	-	-	0.1068	-	-

Table 4: Present the outcomes of the Goodness-of-Fit Statistics for the second Dataset

Model	$-\ell$	AIC	W^*	A^*	$RANK$
TR-ED	-597.1238	-1188.2480	0.0579	0.4199	1
TE-ED	58.2181	122.4361	0.0793	0.5312	4
B-ED	58.0949	122.1899	0.0786	0.5270	3
K-ED	-57.6372	121.2745	0.0617	0.4257	2
ED	58.2186	118.4372	0.0788	0.5284	5

Tables 1 and 3 displays the MLEs of the parameters of the TR-ED and that of the competing distributions obtained from existing literature while, in Tables 2 and 4 we noticed that the proposed distribution (TR-ED) possessed the least values of the goodness of fit statistics and therefore is considered as the best model for the two datasets in this research.

8. Conclusion

A brand-new two parameters family of probability distributions named Transmuted Rayleigh-X was developed using Transmuted Rayleigh distribution as the variate. Consequently, the essential statistical characteristics of the TR-X class of distributions have been determined and comprehensively presented. Definitions for various aspects, including the distribution function, PDF, complimentary CDF, hazard function, ordered statistics, and asymptotic behaviour were provided in a clear and thorough manner. Thus, both CDF and PDF of the TR-X were likewise expanded to ease computation of some features of the TR-X family as they belonged to the Exponentiated- G class. The MLE approach was adopted in estimating the family's parameters. In comparison to TE-ED, B-ED, K-ED and ED a new sub-model (TR-ED) that was generated from the TR-X class has proven to be a better fit when fitted to two different dataset.

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