BENIN JOURNAL OF
STATISTICS
ISSN 2682-5767
Vol. 3, pp. 142– 159 (2020)

Markov Chain applied to Returns on Stock Prices

E. B. Nkemnole[1]* and S. N. Okafor[2]
[1,2]*Department of Mathematics, University of Lagos, Akoka, Nigeria*

Abstract. Markov chain models are commonly used in stock market analysis, manpower planning, and in many other areas because of its efficiency in predicting long run behaviour. This study proposes a simple predicting tool to forecast the future behaviour of stock prices. The study analyses the stock price model using Akaikes Information Criterion (AIC) and Bayes Information Criterion (BIC). The analysis shows that the AIC supports a second order model whereas the BIC supports a first order model. The maximum likelihood function was found to be of the second order. Apparently, the stock price model is time dependent and time homogeneous and thus, best forecasted using a second order model. As described by the GARCH model, the presence of time varying conditional volatility of stock prices, and a lasting effect of today's shock on forecast were found. These results agree with the Markov model based on the AIC. The returns were explored for both short and long run behaviours and it was found that if there is a transition between the states of the return in a current week, the expected return would be higher than the overall average return and this would be realized within the next one week, but if the return is in a stable state then the expected return may move above the overall average return after two weeks.The result also shows that in the long run, the stock price is more stable, and the stock return has a higher probability of remaining in an upward state than in a downward state. This paper uniquely contributes to the literature by demonstrating that NSE series can be modelled as a three-state movement - the upward state, the stable state, and the downward state. This method could help investors save time and make optimal decisions.

Keywords: Markov chain, Nigeria Stock Market, All Share Index, limiting distribution, GARCH.

Published by: Department of Statistics, University of Benin, Nigeria

1. Introduction

The stock market is highly popular among retail and institutional investors. It is a leading investment destination because of its high returns (Kuo, Lee and Lee, 1996; Hassan and Nath, 2005). Equities are arguably one of the most profitable asset classes and constitute the third largest asset class globally, with a total outstanding value of approximately 70 trillion behind securitised debt (approx. 100 trillion) and real estate (approx. 228 trillion), (Savills, 2019). However, the unpredictability of the stock market, evident by its highly volatile nature, especially during periods of uncertainty - such as the Coronavirus pandemic and commodities market selloffs being experienced in recent times - poses some threats to investments in the stock market. Over the past decades, these threats have compelled researchers, both in industry and academia, to consistently undertake studies to devise techniques and procedures for understanding the dynamics and peculiarities of the stock market and mitigating the deleterious effects resulting from volatility, seasonality and dependence on time as well as unusual time-varying correlation surprises with other markets.

Finding a single method that can produce optimal forecasts for all markets could help investors save time and resources and make better decisions, Mallikarjuna and Rao (2019). Some of these techniques include, among others, market timing measures, momentum modelling, stationarity testing, cointegration detection and patterns recognition. For a detailed study on stock market modelling techniques, see the seminal work of Roberts (1959). The legal platform upon which these transactions are carried out is referred to as stock exchange. In the stock market, shares are sold to individuals and groups by companies, in form of stocks. This makes the stock market popular. The general increase in prices of stocks in the stock market is an indication of economic progress being made by the country where the market is domicile.

Generally, there is an association between share prices and investment: an increase in share price would mean an increase in investment while a decrease in share price would mean a decrease in investment. The

*Corresponding author. Email: enkemnole@unilag.edu.ng

more these companies make profit, the more they pay out to shareholders, and less if otherwise. The aim of investment for an investor is to always make profit, but this hardly happens because a financial market is characterized with uncertainties, that may arise from natural calamities, global trends, socio-political policies which may have unprecedented impact on the demand and supply of stocks. According to Shah et al (2019), the market behaves like a voting machine in the short term, but acts like a weighing machine in the longer term, thereby providing a scope for predicting the market movements for a longer timeframe. Zhong and Enke (2017) argued that stock markets are affected by many highly interrelated factors that include economic, political, psychological, and company-specific variables. Thus, an investor's success or failure rate will largely depend on their knowledge of stock market and strategies adopted in predicting the movement of stock prices.

Methods of predicting stock price include regression, time series techniques, data mining, hidden Markov chain, weighted Markov chain, and Markov chain, etc. McQueen and Thorley (1991) applied Markov chain test to annual real and excess returns of the equally weighted and value-weighted portfolios of all stocks in the New York Stock Exchange in periods covering 1947 to 1987. A two state (up-down) Markov chain model was adopted and assumed that the transition probability matrix is stationary throughout the period of analysis, and tested random walk against second-order dependence. The prediction of possible states of the market is more complicated due to the inherent stochastic behavior of stock market (Bhusal, 2017). In Guglielmo and Blasis (2020), first- and second-order price-dividend ratios were computed to show that a different price-dividend ratio is attached to each combination of states of the dividend growth process of each stock.

This paper is a research in line with McQueen and Thorley (1991). The paper undertakes analytical and empirical evaluation of the time dependence and time homogeneity properties of the Markov chain and explores the short and long run behaviours of the stock returns. The remainder of this study is organized as follows. Previous studies on Markov chain based statistical tests and Generalized Auto Regressive Conditional Heteroscedasticity (GARCH) applications to stock markets are reviewed in Section 1. A detailed framework to use this methodology (state space discretization, testing time dependence and testing time homogeneity) is presented in Section 2. Section 3 reports the implementation of the Markov chain and the tests of time dependence and homogeneity, while section 4 concludes the paper.

2. Review of past works

Markov chain based tests have been frequently used in the analysis of statistical properties of various financial time series. For instance, bootstrap procedures in Wang and Scott (1989), and likelihood ratio test and MCMC simulations in Tan and Yilmaz (2002) were used to test for time homogeneity and time dependence. The test allows for nonlinear temporal dependence of stock returns in place of random walk when order of dependence is likely to be greater than one. Markov chain time dependence test requires that the Markov chain associated with the time series under investigation is time homogenous. That is, it is required that the state transition probabilities do not change over time. Though time homogeneity is tested as well, testing time-dependency together with time-homogeneity introduces some challenges for Markov chain based tests.

Niederhoffer and Osborne (1966), Fielitz and Bhargava (1973), Fielitz (1975) and McQueen and Thorley (1991) applied Markov chain test to individual stock prices as well as market indices at various frequencies for different time periods. Many empirical works arrived at a conclusion, against the random walk hypothesis. Some of these studies applied high frequency data such as daily returns, weekly returns and intra-day ticker price changes. Furthermore, they also used large sample properties.

In McQueen and Thorley (1991), the Markov chain test was used to examine both the annual real and excess returns of the value weighted and equally weighted portfolios of total stocks in the New York Stock Exchange between 1947 and 1987. Having adopted a two-state Markov chain, the study assumed stationarity throughout the study period, and used second-order dependence to test for random walk. The study found evidence in favour of mean-reversion in long run stock returns. Felitz and Bhargwa (1973) used daily and weekly returns of 200 observed stocks spanning a period of six years to test for (time homogeneity) and order of dependence of a 3-state Markov chain. The outcome of their result suggests that the observed stocks cannot be handled as a single vector generated Markov chain. Furthermore, they found that the individual stock returns behavior is not influenced by the stationary Markov chain process. The transition probability matrix varied for the observed six-year period.

Niederhoffer and Osborne (1996) used Markov chain test to analyse high frequency data, ticker price change between transactions of seven stocks of the Dow Jones industry average for a 22-day trading period in the month of October 1964. The ticker price changes were split into seven states, and the distance between each state was 1/8th of a dollar. The transaction price changes formed a stationary transition matrix for the seven stocks. The authors rejected random walk after applying a first-order Markov chain, which continually showed that price change reversal is possible than continuity of the change. The result of McQueen and Thorley (1991)is stronger on evidence of mean-reversion than that of Poterba and Summers (1988),and Fama and French (1988).

The literature so far on Markov chain tests gave evidence against random walk, but none of the studies in its totality considered the assumptions of Markov chain. Most of the studies adopted Markov chain test and applied it to their data without examining the properties of the stock returns data in use. Markov chain has advantage over other methods of random walk behaviour by exhibiting the ability to detect likely changes and behavioural patterns of stock index returns over time. A structural break in the time series behaviour of stock returns index would cause rejection of the stationarity of the transition matrix. In order to validate a Markov chain, it is pertinent to test for stationarity of the transition probability matrix.

According to Tan and Yilmaz (2002),a relationship cannot claim stationarity over time without proper testing even though random walk against a first order or higher order return dependence alternatives is rejected. There will be no predictive power for the Markov chain if the transition probability matrix is not stationary over time. Therefore, a proper method would involve subjecting subsample periods to stationarity test in order to ascertain whether the probability matrices are stationary. It is appropriate to test for order of dependence as well for the subsamples when test for stationarity is not rejected.

McQueen and Thorley (1991) assumed stationarity and carried out random walk versus second order dependence test. They used a two-state Markov chain without exploiting all possible options of having more states. Fielitz and Bhargava (1973) and Fielitz (1975) tested for random walk and rejected it in favour of first or higher dependence.

Tan and Yilmaz (2002) introduced a complete testing procedure for Markov chains. They observed a one-to-one communication between the AR (1) parameter and the transition probability matrix of the Markov representation of the series, and then used the result to see if there was a structural break in the data by analysing the statistical power of the Markov chain. The MCMC experiment was also applied to test the finite-sample properties of the Markov chain time dependence and homogeneity test by considering the circular dependence on each other. After testing the null hypothesis of an i.i.d random walk, the empirical size of the Markov chain time dependence test was close to its nominal value regardless of the size or number of states.

In the stock example of Hillier and Lieberman (1995, pp 632), we find the following relationships: (1) state 0: stock increased both today and yesterday in Hiller and Lieberman, by definition is the same as Movement U to U in our term; (2) state 1: the stock increased today and decreased yesterday in Hiller and Lieberman, by definition is the same as D to U in our term ; (3) state 2: the stock decreased today and increased yesterday in Hiller and Lieberman, by definition is the same as U to D in our term; (4) state 3: the stock decreased today and decreased yesterday in Hiller and Lieberman, by definition is the same as D to D in our term. We also noticed that the description of states in Hillier and Lieberman covers the entire states adopted in this study.

Engle (1982) asserts that modelling volatility through Auto Regressive Conditional Heteroscedasticity (ARCH) is sufficient to capture the random nature of conditional variance using lagged disturbance. Bollerselev (1986) developed a more generalized model called the Generalized Auto Regressive Conditional Heteroscedasticity (GARCH) model which is appropriate if an auto regressive moving average (ARMA) is assumed for the error variance.

The study is designed to extend the works of Tan and Yilmaz (2002) and compare the obtained results with the GARCH model of the volatility of stock prices in the Nigerian Stock Market.

3. Materials and methods

3.1 *Markov chains model*

A Markov chain is a sequence of random variables $X_0, X_1, X_2, X_3, \ldots$, with the Markov property namely that, the conditional probability of any future event, given any past event and the present state, is independent of the past event and depends only on the present state. Formally, $X_0, X_1, X_2, X_3, \ldots$, is a Markov chain if it satisfies the Markov property, equation (1):

$$P(X_{t+1} = s|X_t = s_t, ..., X_0 = s_0) = P(X_{t+} = s|X_t = s_t) \quad (1)$$

$\forall$ t= $1, 2, 3, ...$ and $\forall$ states, $s_0, s_1, s_2, \ldots, s_t, s$.

DEFINITION 3.1 *The state of a Markov chain at time t is the value of X_t For instance, if $X_t = 5$, it means the process is in state 5 at time t. Definition: The state space of a Markov chain, S, is the set of values that each X_t can take. For example, $S = 1, 2, 3, 4, 5$. S has a size N, or possibly an infinite size.*

3.1.1 Transition probabilities

In a Markov chain, the conditional probabilities: $P(X_{t+} = j|X_t = i)$ are called transition probabilities. The transition probability matrix is the matrix obtained from transition frequency table. It is an array of $p_{ij's}$, the probability of transitioning from state i at time n to state j at time $n + 1$.

$$[P] = \begin{pmatrix} P_{11} & P_{12} & \ldots & P_{1n} \\ P_{21} & P_{22} & \ldots & P_{2n} \\ P_{31} & P_{32} & \ldots & P_{3n} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ P_{n1} & P_{n2} & \ldots & P_{nn} \end{pmatrix}$$

3.1.2 Distribution of X_t

Let $(X_0, X_1, X_2, X_3, \ldots)$ be a Markov chain, having a state space $S = 1, 2, \ldots, N$. Since each X_t is a random variable, it has a probability distribution. The probability distribution of X_t is an N*1 vector. Consider X_0. Let

$$\pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ . \\ . \\ . \\ \pi_N \end{pmatrix} = \begin{pmatrix} P(\pi_0 = 1) \\ P(\pi_0 = 2) \\ . \\ . \\ . \\ P(\pi_0 = N) \end{pmatrix}$$

So $X_0 \sim \pi^T$ means that the row vector of probabilities is given by the row vector π^T.

THEOREM 3.2 *Let $(X_0, X_1, X_2, X_3, \ldots)$ be a Markov chain with an N x N transition matrix P. Suppose the probability distribution of X_0 is given as a 1 x N row vector π^T, then the probability distribution of X_t is given by the 1 x N row vector $\pi^T P^T$. So, $X_0 \sim \pi^T : X_t \sim \pi^T P^T$.*

Proof. ForX_1, using the partition rule and conditioning on X_0: $P(X_1 = j) = \Sigma_1^N P(X_1 = j|X_0 = i)P(X_0 = i) = \Sigma_1^N P_{ij}\pi_i$ = $\Sigma_1^N \pi_i P_i j = (\pi^T P)j$. For X_2, using the partition rule and conditioning on X_0, we have $P(X_2 = j) = \Sigma_1^N P(X_2 = j|X_0 = i)P(X_0 = i) = \Sigma_1^N (P^2)_{ij}\pi_i = (\pi^T P^2)j$. Continuing in the same way as before, we have$X_t \sim \pi^T P^T$ (Fewster, 2014). ■

3.1.3 n-step transition probability

The Chapman Kolmogorov equation

For any $n \geq 0, m \geq 0, i \in S, j \in S$, $P_{ij}^{n+m} = \Sigma_{k \in s} P_{i,}^n P_{k,j}^m$, this is obtained by taking into account the state of the chain at time n: Given $X_0 = i, P_{i,k}^n = P(X_n = k|X_0 = i)$ is the probability that the state at time n is k. However, if $X_n = k$, the future after time n, is independent of the past. Therefore, the probability of the chain being at state j, m time units later (n+m) would be P_{kj}^m, yielding the product ,$P_{i,k}^n P_{kj}^m = P(X_n = k, X_{n+m} = j|X_0 = i)$.

By way of generalization, P^n_{ij} produces the n-step transition probabilities which is the conditional probability that the process will be in state j after n-steps provided that it starts in state i at time t. The n-step transition probabilities are defined as the conditional probability.

$$P(X_{t+n} = j|X_t = i) = P(X_n = j|X_0 = i) \tag{2}$$

$\forall\, t = 0, 1, ...$

Proof. See Sigman(2009). ■

3.2 *Time homogeneity and time dependence test*

3.2.1 *Homogeneous Markov chain*

In a time homogeneous Markov Chain, the transition probabilities are independent of the time index t: $P(X_{t+1} = j|X_t = i) = P(X_1 = j|X_0 = i)$ for all $t = 0, 1, \dots$. By this, we hope to test the order using the entire data set, and then partition the entire data set into sub intervals and test for time dependence as well as time homogeneity. The test statistic is: $-2ln(\Lambda) = 2\Sigma_r\Sigma_{i,j}n^r_{i,j}[ln(p^r_{i,j}) - ln(p_{i,j})]$, $i\epsilon S^u, j\epsilon S, r = 1, 2, 3, ..., R$

3.2.2 *Time dependent test*

Supposing a Markov chain of order u is $[I_t, t = 0, 1, 2, 3, ...]$ such that

$P(I_t = i_t|I_0 = i_0, I_1 = i_1, ..., I_{t-1} = i_{t-1}) =$

$$P(I_t = i_t|I_{t-u} = i_{t-u}, I_{t-u+1} = i_{t-u+1}, ..., I_{t-1} = i_{t-1}) \tag{3}$$

Being a special case for n=1, we refer $[I_t, t = 0, 1, 2, 3, ...]$ as a first order Markov chain or just Markov chain provided

$$P(I_t = i_t|I_0 = i_0, I_1 = i_1, ..., I_{t-1} = i_{t-1}) = P(I_t = i_t|I_{t-1} = i_{t-1}).$$

Consider another special case when the Markov chain is of order 0. That is, if $[I_t, t = 0, 1, 2, 3, ...]$ is independent or random, the transition in different times are independent process if $\forall\, t, t = 0, 1, 2, 3, ...,$

$$P(I_0 = i_0, = I_1 = i_1, ..., I_t = i_t) = P(I_0 = i_0)P(I_1 = i_1)...P(I_t = i_t)$$

Our approach in this study is to set the order to zero and then estimate the state transition probabilities for order 0 and order 1. Consequently, we test the hypothesis that the Markov chain is order 0 against the alternative hypothesis that the Markov chain is not order 0. If this hypothesis is not rejected, then we reach a conclusion that the order is 0 and proceed with the time homogeneity test. If the hypothesis is rejected, then, we raise the current order by one until an allowed limit is reached and we repeat the same procedure. It is necessary for us to set a limit to the maximum order to be tested. This is because as the order increases, the number of parameters to be estimated increases exponentially, thereby causing the testing power to decrease (Jonsson, 2011; Tan and Yilmaz, 2002).

In this work, stock prices are modelled as a three-state Markov chain. In addition, the transition probabilities are restricted based on the random walk hypothesis (McQueen and Thorley (1991). We explore and separate our stock data into three states and apply Markov chain methodology to the yearly stock data as earlier stated above. Let X_n be the observed stock price at time n, and $\bar{X}$ the overall mean of the series. Let I_t be given as: $I_t = U$ if $X_n > \bar{X} + kMAD$, $I_t = S$ if $\bar{X} + kMAD \geq I_t \geq \bar{X} - kMAD$, and $I_t = D$ if $X_n < \bar{X} - kMAD$. The corresponding state space, Ω=U,S,D where U denotes the upward movement of stock price at time n, S denotes stable movement of stock price at time n, and D denotes the downward movement of stock price at time n. The mapping of the three states was arrived at using the

Mean, the Mean Absolute Deviation (MAD) of the stock price, and an assumed constant, k=0.5. This choice of value for k helps to yield approximate equiprobable regions. The representation considers both the direction of movement and the magnitude of these changes. The test statistic for the likelihood ratio test is: $-2ln(\Lambda) = 2\Sigma_{i,j}^{v}[ln(p_{i,j}^{v}) - ln(p_{i,j}^{\sim u})]\ i\epsilon S^{v}$to $j \in S$.

3.3 *Estimating state transition probabilities*

A time homogeneous Markov chain of order=u has the following characteristics using its state transition matrix. $P = (P_{ij})$:

$$P_{ij} = P(I_t = j|I_{t-u} = i_{t-u}, I_{t-u+1} = i_{t-u+1}, ..., I_{t-1} = i_{t-1})$$

where $i\epsilon S^u$, $j\epsilon S$, $t = 0, 1, 2, 3, ...$ If the order of the chain exceeds one state, then i includes more states, e.g. for a second order Markov chain defined on the state space [S,D],$i\epsilon$ [SD,DS,DD] and $j \in$ [S,D]. If we can determine that the state transition probabilities changes not, that is, time homogenous for a specified period, and the order is established too, then we can estimate the transition probabilities from the observed movements using the maximum likelihood method(MLE).

$$\hat{p}_{ij}^{MLE} = \frac{n_{ij}}{\Sigma_{u=1}^{k} n_{iu}}, i \in S^u, j \in S.$$

3.3.1 *Establishing the order of a Markov chain*

We test the null hypothesis that the Markov chain is of order u against the alternative that it's of order v,$v > u$. We let the time homogenous state transition matrix of a u^{th} order Markov chain be: $P^u = p_{ij}^u$, where $p_{ij}^u = P(I_t = j|I_{t-u} = i_{t-u}, I_{t-u+1} = i_{t-u+1}, ..., I_{t-1} = i_{t-1})\ i\varepsilon S^u, j\varepsilon S, t = 0, 1, 2, 3, ...$ in the same manner,let the time homogeneous state transition matrix of a v^{th} order of the Markov chain be given as $P^v = p_{ij}^v$, where $p_{ij}^v = P[I_t = j|I_{t-v} = i_{t-v}, I_{t-v+1} = i_{t-v+1}, ..., I_{t-1} = i_{t-1}]\ i\varepsilon S^v, j\varepsilon S$,$t = 0, 1, 2, 3, ...$ Let n_{ij}^v represent total number of transitions observed from state $i\varepsilon S^v$to $j\varepsilon S$. Then asymptotic likelihood ratio test statistics is given as:

$$-2ln(\Lambda) = 2\Sigma_{i,j}^{v}[ln(p_{i,j}^{v}) - ln(p_{i,j}^{\sim u})],\ i\varepsilon S^v \text{ to } j\varepsilon S,$$

where $p^{\sim u}$= $np_{i,j}^{\sim u}S = [P^u, P^u, P^u, ..., P^u]^T$ and $[P^u, P^u, P^u, ..., P^u]$ corresponds to the sequence of 2^{v-u}. The χ^2 test corresponding to the test statistic is distributed with $(n_s^v - n_s^u)(n_s - 1)$ degrees of freedom, where n_s is number of states(Tan and Yilmaz,2002).

The procedure above is employed to test the independence of transition probabilities against the alternative hypothesis that the Markov chain is of order v, which is similar in testing order 0 against order v. Recall that a 0^{th} order model means that the variables, X_i are independent, that is $P(X_j/X_1, X_2, \ldots, X_{j-1}) = P(X_j)$.

3.4 *Testing time homogeneity*

A Markov chain is said to be time homogeneous if the state transition probabilities do not change with time. In this study, we have only one observation of time series, and to test for time homogeneity, we need to divide the time series observation into subintervals.Time homogeneity otherwise known as stationarity is referred to as a situation where the transition probabilities for each sub intervals are the same or identical. Since we have only one time series observation, the process begins by first dividing $I_t, t = 1, 2, 3, ..., T$ into r different subintervals of equal length and test if the estimated transition probabilities for each period are statistically different from the estimated transition probabilities for the full period. We let the state transition probability of a u^th order Markov chain relating to period $r, r = 1, 2, 3, ..., R$ be expressed as:

$$p_{i,j}^r = P(I_t = j|I_{t-v} = i_{t-v}, I_{t-v-1} = i_{t-v-1}, ..., I_{t-1} = i_{t-1}),$$

$t\varepsilon[(r + 1)\Delta, r\Delta]$,where $\Delta = [\frac{(T + 1)}{R}]$.

We test the hypothesis that there is no significant difference between the state transition matrix, at step r, P^r,for each period and the transition matrix for the whole period P against the alternative hypothesis of there is a difference. The asymptotically equivalent test statistic (Tan and Yilmaz, 2002) to the likelihood ratio test is expressed thus,

$$2ln(\Lambda) = 2\Sigma_r \Sigma_{i,j} n^r_{i,j} [ln(p^r_{i,j}) - ln(p_{i,j}) \tag{4}$$

$i \epsilon S^u, j \epsilon S, r = 1, 2, 3, ..., R$ having a χ^2 test distribution with $(R - 1)n_s(n_s - 1)$ degrees of freedom.

According to Tan and Yilmaz (2002) if this test is rejected, the process cannot be assumed to be time homogeneous. Hence, no other tests in respect of the order behavior can be carried out by using a single transition matrix estimated from the observed transitions. Similar test can be applied to each sub period by dividing each sub period further into smaller intervals.

If the number of observed transitions in each sub period is small, then it is more plausible to reject the null hypothesis against the alternative hypothesis chiefly as a result of lack of data. The test may result in any of the following three different conclusions (Tan and Yilmaz, 2002). Firstly, the test estimates an order (lower than the maximum allowed) and accepts the null of time homogeneity. Secondly, where time homogeneity is rejected, the order of dependence cannot be estimated with Markov chains for the sample considered. Finally, it is possible for the test to reject all orders of dependence equal to or less than the maximum order initially specified by the researcher, then it is considered as inconclusive, as it is not possible to conclude about the order, and so cannot carry-on with the time homogeneity test.

3.5 *Estimating expected returns*

The expected short and long run returns are estimated using the following formula. For the long run,

$$E_R = \pi_j \theta_i \tag{5}$$

For the short run,

$$E_r = P^n \theta_i \tag{6}$$

where E_R is the expected long run of the return, E_r is the expected short run of the return, π_j is the steady state probability, θ_i is the mean returns for state i while P^n is the probability at the n-step.

Let W_n represent the closing price for the n^{th} week, and W_{n-1} be the closing price for the previous week. We define another stochastic variable, Z_n,as the change in prices between the current week and the previous week in the stock market. This is given as: $Z_n = \dfrac{(W_n - W_{n-1})}{W_{n-1}}$. A positive Z_n implies a gain on investment while a negative Z_n implies a loss on investment. Similarly as above, we let I_t be given as: $I_t = U$ if $Z_n > \bar{Y} + kMAD$; $I_t = S$ if $\bar{Y} + kMAD \geq Z_n \geq \bar{Y} - kMAD$; $I_t = D$ if $Z_n < \bar{Y} - kMAD$, where $\bar{Y}$ is the average return for the series and $K = 0.5$ is a constant. Thus we have a three state transition matrix as before.

3.6 *Generalized ARCH (GARCH) Model*

According Bollerslev (1986), the conditional variance for GARCH (p, q) model is expressed generally as:

$$\sigma_{t^2} = \omega + \Sigma^q_{i=1} \alpha_i \varepsilon^2_{t-i} + \Sigma^p_{i=1} \beta_j \sigma^2_{t-j} \tag{7}$$

$$= \omega + \alpha(L)\varepsilon^2 + \beta(L)\sigma^2_t$$

where p is the order of the GARCH terms, and q is the order of the ARCH terms, where $\omega > 0; \alpha_i \geq 0; i = 1, \ldots, q - 1; j = 1, \ldots, p - 1$ and $\alpha_q, \beta_p > 0$. σ^2_t is the conditional variance and ε^2 the disturbance term. $\Sigma^q_{i=1} \alpha_i = \alpha(L)$, and $\Sigma^p_{i=1} \beta_j = \beta(L)$. $\alpha(L)$ and $\beta(L)$are lag operators: $\alpha(L)) = \alpha_1 L + \alpha_2 L^2 + \alpha_3 L^3 +$

$... + \alpha_q L^q$ and $\beta_1 L + \beta_2 L^2 + \beta_3 L^3 + ... + \beta_p L^p$. When p = 0, the process reduces to an ARCH(q), and when $p = q = 0, \varepsilon_t^2$ is just a white noise. The reduced form of (7) is the GARCH (1, 1). The variance equation of the GARCH (1, 1) model can be expressed as

$$t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{8}$$

$\omega, \alpha_1, \beta_1$ are non-negative while $\alpha_1 + \beta_1 < 1$ to achieve stationarity.

4. Results and discussion

4.1 *Data*

The method discussed in the previous section is applied to model the Nigerian Stock Market (NSE) III-Share Index data from 1st of August 1, 1999 to January 19, 2018. The data is a weekly data collected at the end of the trading week and covers about 992 observations. The source of the data is www.bloomberg.com. Figure 1 shows the movement of stock prices from January 1999 to January 2018. The market experienced a downward trend in stock price between 1999 and 2001, then a stable rise from 2001 to 2006. Then it continues to rise between 2001 and 2008, followed by a stable fall between 2008 and 2013, and continues in a sinusoidal pattern in the subsequent years. Table 1 reports the descriptive statistics.

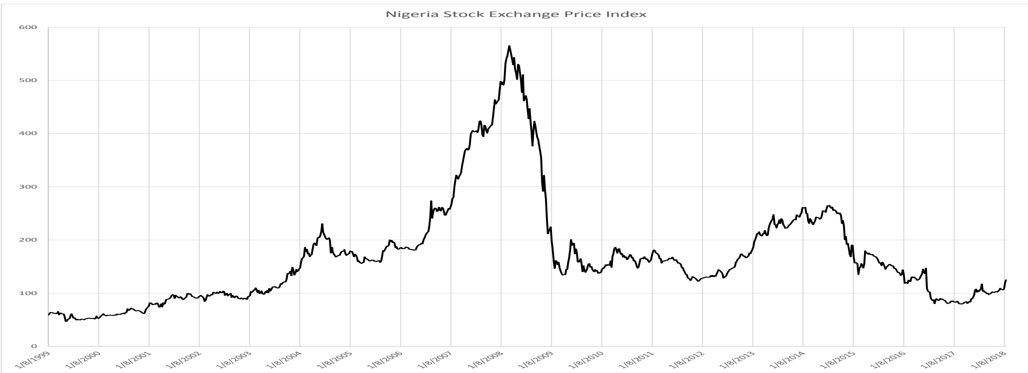

Figure 1: Stock series

Table 1: Descriptive statistics summary for the NSE index

Mean	171.0322
Standard Error	3.223652
Median	154.355
Mode	49.87
Standard Deviation	101.5322
Sample Variance	10308.8
Kurtosis	2.99427
Skewness	1.684732
Range	518.51
Minimum	46.92
Maximum	565.43
Sum	169663.9
Count	992

The positive skewness shows that the upper tail of the distribution is thicker than the lower tail, implying that the stock prices rise more of the time than they fall, an indication of good news for the investors. The standard deviation value shows a high level of volatility in the stock market.

4.2 *Estimation of transition probability matrix*

Table 2: Transition frequency table of the NSE index

State	D	S	U	Total
D	146	4	0	150
S	3	710	8	721
U	0	8	113	121

Table 2 shows the transition frequency table of the NSE index where D denotes Downwards movement, S represents stable movement and U stands for Upwards movement of stock price. There are 146 entries in the first row and this is the total number of the observed stock prices that remained in the downwards state from week to week. The entry 4 is the total number of observed stock prices in the downwards state in the week considered and this moved to stable state in the subsequent weeks. No observation was recorded for prices that transits from the downwards state this week to the upwards state in the following weeks. Other rows are similarly explained.

The corresponding transition probability matrix is:

$$P = \begin{pmatrix} 0.973333 & 0.026667 & 0 \\ 0.004161 & 0.984743 & 0.011096 \\ 0 & 0.066116 & 0.933884 \end{pmatrix}.$$

The first row in the matrix signifies that if the price is in Down state, the next week's price will fall (D), remain stable (S) or rise (U) by 97.33%, 2.67% or 0.00% respectively. The second row indicates that if the price is in stable state today, then the probability of it falling, remaining stable or rising next week is 0.42%, 98.47% or 1.11% respectively. Again, the third row indicates that if the price is in Up state today, then the probability of it falling, remaining stable or rising next week is 0.00%, 6.66% or 93.34% respectively.

We can also observe that the transition matrix is transient and in a reducing class and the steady state probability is π = $[0.1178639, 0.7553659, 0.1267702]$ which means that 11.79$ of the time, the stock price will be in Down state, 75.53% of the time in stable state, and 12.68% of the time in Up state in the long run.

4.3 *Testing the order of a Markov chain*

According to Tan and Yilmaz (2002), the asymptotic likelihood ratio test statistics is given as

$$-2ln(\Lambda) = 2\Sigma_{i,j}^{v}[ln(p_{i,j}^{v}) - ln(p_{i,j}^{\sim u})], \qquad i \in S^v \text{ to } j \in S$$

with a $\chi 2$ distribution of $(n_s^v - n_s^u)(n_{s-1})$ degrees of freedom, where n_s is number of states.

Hypothesis. H_0: The Markov chain is of order 0 (the variables are independent) versus H_1: The Markov chain is not of order 0.

If our hypothesis is rejected, we will test a higher order which also means that the number of our parameters to be estimated will increase exponentially. Therefore, to avoid the weakening of our testing power we will set our maximum higher order to be tested as order 2.

Now, $2\Sigma_{i,j}^{v} n_{i,j}^{v}[ln(p_{i,j}^{v}) - ln(p_{i,j}^{\sim u})] = 1313.44$, which alternatively can be simplified as $2\Sigma_{i,j}^{v} n_{i,j}^{v}[ln(p_{i,j}^{v}) - ln(p_{i,j}^{\sim u})] = 2[L^v - L^u]$, where $L^v = -111.28$ is the likelihood of v, and $L^u = -768$ is the likelihood of u. The value -111.28 is the most likely parameter value that could generate the observed data for order v. Similarly, -768 is the most likely parameter value that could generate the observed data for order u. Therefore, $2[L^v - L^u] = 1313.44$ as above. $n_s = 3, n^v = 2^1, n^u = 2^0$. $(n_s^v - n_s^u)(n_s - 1) = (2-1)(3-1)$. Therefore, $\chi^2_{(2)} = 0.102587$. Since $1313.44 > 0.102587$, we reject Ho and conclude that the order is order 1. We rejected order is order 0, we proceed to test a higher order, this time, order is order 1 by first estimating the transition probability matrix of order 2.

Table 3: Transition frequency table

State	D	S	U	Total
DD	152	2	0	154
DS	0	4	0	4
DU	0	0	1	1
SS	3	700	6	709
SD	2	0	0	2
SU	0	3	2	5
UU	0	0	109	109
UD	0	0	1	1
UU	0	5	1	5

order v	log likelihood
0	-768
1	-111.28
2	-70.71

4.3.1 Estimation of transition probability matrix for order 2

The corresponding transition probability matrix is given as

$$P = \begin{pmatrix} 0.987013 & 0.012987 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0.004231 & 0.987306 & 0.008463 \\ 1 & 0 & 0 \\ 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0.833333 & 0.166667 \end{pmatrix}$$

Hypothesis. H0: The Markov chain is of order 1 versus H1: The Markov chain is not of order 1.

Now, $L^v = -70.71$ and $L^u = -768$. So $2[L^v - L^u] = 1394.58$ Therefore, $\chi^2_{(12)} = 5.2603$. Since $1395 > 5.2603$, we reject Ho and conclude that the order is greater than order 1.

4.4 *Model comparison*

We compute the Akaike's Information Criterion (AIC) and the Bayesian Information Criterion (BIC) for model comparison. These methods use the likelihood ratio statistics and modify it by a penalty term. The preferred model is the one with the lowest value. According to Singer, P (2013), we define both methods (v order, u order, s number of states, n observations) as:

$$AIC(v) = v\eta u - 2 * (S^v - S^u) * (S - 1)$$

$$BIC(v) = v\eta u - (S^v - S^u) * (S - 1) * ln(n).$$

Seeing the likelihood values, the most obvious assumption would be to accept order two since it has the highest value of likelihood. However, with our log likelihood estimations above, we can set $v = 2$ and test if we should perhaps prefer a lower order model even though the higher order chain has higher log likelihood values. So we proceed as follows. $v\eta u = -2 * (LL_v - LL_u)$, where LL denotes log likelihood Table 5 reveals that in the AIC column, order 1 has the lower value and should be preferred. Whereas, for the BIC column, order 0 has the lower value and should be preferred. The negative values obtained in the AIC and BIC columns creates no problem since only the relative values matter and not the absolute value. So,by adding an arbitrary constant M to the information criteria values, say M=1500 the results are still the same,

Table 4: Estimation results – model comparison

order v	$v\eta u$	AIC	BIC
0	1313.44	1305.44	- 1304.54
1	-82.56	-106.56	-248.15

Table 5: Re-scaled values

order v	AIC	BIC
0	2805.44	195.46
1	1393.44	1251.85

Table 6: Transition frequency table fork_1

State	D	S	U	Total
D	122	2	0	124
S	1	277	5	283
U	0	4	85	89

Table 7: Transition frequency table for k_2

State	D	S	U	Total
D	25	2	0	27
S	2	433	3	438
U	0	4	27	31

see (Anderson, 2008 ; Burnham and Anderson,2004). After adding 1500 to the Information Criteria values, Table 6 produced the same outcome as before in Table 5- order 1 is preferred with AIC,and order 0 with BIC.

4.4.1 *Test of independence of transition probabilities*

This test is equivalent to testing order 0 vs higher orders using the above procedures. That is in our test of hypothesis:

H_0:The transition probabilities are independent$\equiv$ The Markov chain is of order 0.

H_1: The transition probabilities are not independent$\equiv$ The Markov chain is not of order 0.

Since we rejected H_0 from our test above, we hence conclude that the Transition Probabilities are dependent.

4.5 *Test of time homogeneity*

Since there is only one available series of data, we need to first divide the entire time series data (992 observations) into K different equal length of observations, to test for time homogeneity. By dividing the original observation into equal halves, we have 496 equal observations. Hence, our $K = 2$, and we obtain the following transition tables and corresponding probability matrices for k_1 and k_2 respectively:

The corresponding transition probability matrix for k_1 is

$$P = \begin{pmatrix} 0.983871 & 0.016129 & 0 \\ 0.003533 & 0.978799 & 0.017668 \\ 0 & 0.044944 & 0.955056 \end{pmatrix}$$

The corresponding transition probability matrix for k_2 is

Table 8: Transition frequency

State	D	S	U	Total
D	229	5	224	458
S	6	0	5	11
U	219	7	297	523

$$P = \begin{pmatrix} 0.925926 & 0.074074 & 0 \\ 0.004566 & 0.988584 & 0.00685 \\ 0 & 0.129032 & 0.870968 \end{pmatrix}$$

Hypothesis. H_0: The state transition matrix of period k_1 is not statistically different from the main transition matrix versus H_1: It is statistically different.

Here, $2\Sigma_k\Sigma_{i,j}n_{i,j}^k[ln(p_{i,j}^k) - ln(p_{i,j})] = -42.70$. The associated Chi-square tabulated is 1.63539. Since $-42.70 < 1.63539$, we accept Ho and conclude that it is time homogenous for period k_1.

In a similar way, we carry out the homogeinity test for period k_2.

Hypothesis. H_0: The state transition matrix of period k_2 is not statistically different from the main transition matrix versus H_1: It is statistically different.

Thus, $2\Sigma_k\Sigma_{i,j}n_{i,j}^k[ln(p_{i,j}^k) - ln(p_{i,j})] = -11.50$. The associated Chi-square tabulated is 1.63539. Since $-11.50 < 1.63539$, we accept Ho and conclude that it is time homogenous for period k_2 as well. Since the hypothesis test for main probability matrix and subintervals accepted the null hypothesis of homogeinity, we conclude that the state transition probabilities do not change with time.

4.6 *Estimation of expected returns*

We obtain the transition frequency below The transition probability matrix is:

$$P = \begin{pmatrix} 0.5 & 0.010917 & 0.489083 \\ 0.545455 & 0 & 0.454545 \\ 0.418738 & 0.013384 & 0.567878 \end{pmatrix}.$$

Similarly, as in the case of stock price, the P Matrix is the probability vector showing the proportional change following the transition in stock return in two consecutive weeks. The first vector in the matrix signifies that if the return is in Down state, the next week's return will fall(D), remain stable (S) or rise(U) by 50% ,1.1% or 48.90% respectively. The second vector indicates that if the return is in Stable state today, then the probability of it falling, remaining stable or rising next week is 54.5%, 0.00% or 44.5% respectively. Again, the third vector indicates that if the return is in Up state today, then the probability of it falling, remaining stable or rising next week is 41.9%, 1.3% or 56.8% respectively We can also observe that the transition matrix is transient and in a reducing class.

4.7 *Limiting probability for the weekly returns*

$$P = \begin{pmatrix} 0.500000 & 0.010917 & 0.489083 \\ 0.545455 & 0.000000 & 0.454545 \\ 0.418738 & 0.013384 & 0.567878 \end{pmatrix}$$

$$P^8 = \begin{pmatrix} 0.4574432 & 0.01209363 & 0.5304632 \\ 0.4574432 & 0.01209363 & 0.5304632 \\ 0.4574432 & 0.01209363 & 0.5304632 \end{pmatrix}$$

The limiting (steady) state probability is $\pi = [0.4574432, 0.01209363, 0.5304632]$. This implies that in the long run, the return will be 45.74% of the time in the Downward state, 1.21% of the time in stable state, and 53.05% of the time in Upward state.The mean of the overall return is found to be 0.001429419 and the calculated mean return of various states are presented in a 1 x 3 vector called θ_i. Hence, the expected long run return are calculated using equation (5):

$$E_R = \pi_j \theta_i$$

where

$$\pi_j = (0.4574432, 0.01209363, 0.5304632)$$

$$\theta_i = \begin{pmatrix} -0.02537 \\ 0.000503122 \\ 0.024916747 \end{pmatrix}$$

The average return for the downward state is negative which implies a loss. The stable state has almost an approximate average return of zero which signifies a break even point between the upward and the downward states. Whereas, the average return for the upward state is positive which implies a profit. It is also clear that the average return of the downward state is approximately equal to the average return of the upward state. Thus, $E_R = 0.001618168$.

Furthermore, the expected short run returns can be realized using equation (6):

$$E_r = P^n \theta_i$$

For n= 1, we have

$$E_r = \begin{pmatrix} -0.000493150 \\ -0.002512411 \\ 0.003533023 \end{pmatrix}.$$

For n=2, we have

$$E_r \begin{pmatrix} 0.001453939 \\ 0.001336927 \\ 0.001766199 \end{pmatrix}.$$

For n=8, we have

$$E_r \begin{pmatrix} 0.001618166 \\ 0.001618166 \\ 0.001618166 \end{pmatrix}.$$

The coefficients of the constant term and the estimated parameters are all positives, and statistically significant save for the constant term (see Appendix). The result shows the presence of time varying conditional volatility of the NSE stock prices and large persistence of volatility shocks, meaning that the effect of today's shock will remain in the forecast for a longer period of time. The Akaike Information Criterion (AIC) is the preferred model over Bayes Information Criterion (BIC), which further gives credence to our Markov chain model as earlier discussed. Also, the shape parameter is significant, buttressing the goodness of fit of the model which is positive and significant for all groups.

Residual diagnostics: Ljung Box test of no serial correlation

Since the corresponding P- values for the residuals are a mix of less than 0.05 and greater than 0.05 (p-values < 0.05 and p-values > 0.05), we conclude that the residuals do not behave completely as a white noise.

Test for ARCH behaviour in residuals

Taking a keen look at the Standardized Squared Residuals test and the ARCH LM Tests, we observe that the p-values are greater than 0.05 (p-values > 0.05) and we do not reject the null hypothesis of non-serial correlation. Hence, we can conclude they are serially correlated.

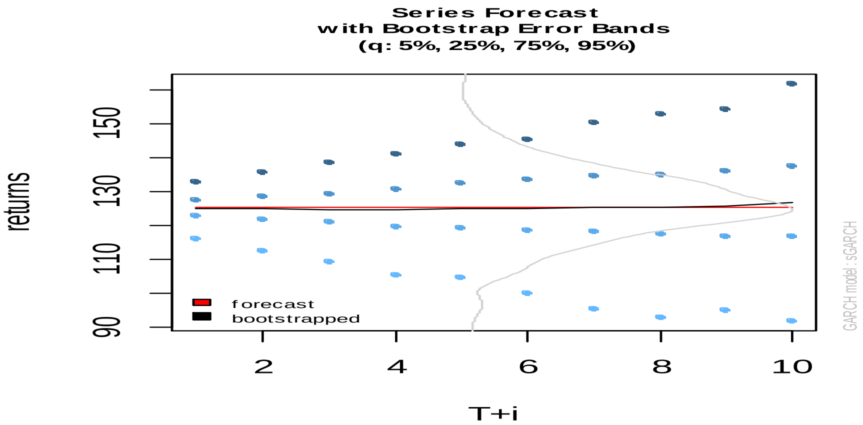

Figure 2: Stock price forecast covering a 10-year period

Table 9: Series (summary)

	Min	q.25	Mean	q.75	Max	Forecast[analytic]
t+1	92.8472	122.37	125.06	127.96	147.55	125.21
t+2	93.1129	120.71	125.66	130.06	166.90	125.21
t+3	80.2379	119.87	125.49	130.77	180.79	125.21
t+4	60.4081	119.50	125.73	132.44	194.49	125.21
t+5	75.1246	119.61	126.50	134.02	206.77	125.21
t+6	3.7764	118.75	126.83	135.11	206.96	125.21
t+7	36.0418	118.71	127.38	136.26	212.90	125.21
t+8	28.5541	118.50	127.99	136.44	230.48	125.21
t+9	25.2364	118.03	128.96	138.20	228.26	125.21
t+10	15.4194	118.19	130.17	139.38	247.97	125.21

5. Conclusion

Stock movement has the capacity to influence growth and business policies. Therefore, it is crucial to effectively model these movements for optimal investment and decision making. This study tests the Markov chain in time dependence and in homogeneity by analyzing the Nigeria Stock Exchange NSE III-Share Index time series data using the framework developed by Tan and Yilmaz (2002). The obtained Markov models were tested for order, time dependence and homogeneity. It was found that the stock price model exhibits a higher order, of which we placed a maximum order on order 2 due to the exponential increase of parameters to be estimated when an order is increased by one. The model was assessed using the Akaike information Criteria and the Baye's information criteria, and consistent with our model, the Akaike information criteria supports a second order model whereas the Baye's information criteria supports a first order model. The Maximum Likelihood function was also estimated and found to be of higher order. Furthermore, we find the

Table 10: Sigma (Summary)

	Min	q.25	Mean	q.75	Max	Forecast[analytic]
t+1	6.4039	6.4039	6.4039	6.4039	6.4039	6.4039
t+2	5.2853	5.3501	6.1462	6.1814	19.9011	6.5074
t+3	4.4127	4.7848	6.0551	6.3390	20.2367	6.6092
t+4	3.7495	4.3541	6.0257	6.5613	26.6837	6.7094
t+5	3.2641	4.1045	6.1027	7.0176	27.4594	6.8080
t+6	2.9120	3.9143	5.9441	6.8420	25.9145	6.9051
t+7	2.7086	3.8061	6.1596	6.9000	52.8585	7.0007
t+8	2.5262	3.5224	6.0680	6.8896	52.1591	7.0950
t+9	2.3538	3.3934	6.0139	6.7735	43.1808	7.1879
t+10	2.3332	3.4208	6.0236	6.6844	71.5841	7.2796

model to be time dependent and time homogeneous. The GARCH model result reveals the presence of time varying conditional volatility of stock prices, and a lasting effect of today's shock on forecast. The diagnostic test shows the presence of serial correlation. A forecast was carried out and the result shows a constant mean with a steady rise of volatility over the ten-year period, which confirms the presence of serial correlation. Moreover, the model also agrees with the Markov model based on the AIC. The returns were also analyzed for short and long run behaviours, and interestingly, we discovered that if the state is currently in a Downward or Upward state this week, a return higher than the overall average return would be realized in just one week and would almost remain positively stable until a steady state is reached after eight weeks. However, when the return is in stable state this week, a return above the overall average return would only be realized after two weeks and also remain positively stable until the steady state is reached in eight weeks. We also noticed, that if the state is currently in Upward state this week, subsequent expected returns will always be above the expected long run return. The result also shows that in the long run, the stock price is more stable, and the stock return has a higher probability of remaining in an upward state than in a downward state. The outcome of these expected returns shows that investing in Nigerian Stock Exchange in both short and long term would yield positive returns most of the times.

This paper uniquely contributes to the literature by demonstrating that NSE series can be modelled as a three-state - the upward state, the stable state, and the downward state. This method could help investors save time and make optimal decisions. The NSE result looks promising on the returns, however, the GARCH modelling shows a presence of high volatility. Therefore, we propose that future study should further divide the series into halves and carry out similar test to ascertain the validity of the model. Future researchers should also employ GARCH model on the log series to eliminate the effects of volatility.

The policy implications of this method is to inform the investment behaviour of major market participants such as the Pension Fund, Insurance and Asset Managers on the dynamics of the market, especially in understanding how and when to enter the market in order to maximize returns and safe guard people's pension and wealth.

Acknowledgement

The authors are deeply grateful to the anonymous reviewers for their great contributions in enhancing the quality of the original manuscript.

References

Anderson D.R.(2008). Model Based Inference in the Life Science: a Primer on Evidence. Springer Science+ Business Media, LLC.

Bollerslev, T. (1986). Generalised autoregressive conditional heteroskedasticity. Journal of Econometrics, **31**: 307-327.

Bhusal, M.K. (2017). Application of Markov chain model in the stock market trend analysis of Nepal. International Journal of Scientific and Engineering Research, **8**(10): 1733-1745.

Burnham K.P. and Anderson D.R.(2002). Model Selection and Multi-model Inference: A Practical Information-Theoretic Approach. Springer-Verlag New York, Inc.

Wang, D. Q. and Scott, D. J. (1989). Testing a Markov chain for independence. Journal of Communications in Statistics- Theory and Methods, **18**(11).

Engle, R.F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. Econometrica, **50**: 987-1007.

Fama, E. F. and French, K. R. (1988). Permanent and temporary components of stock prices. Journal of Political Economy, **96**: 246-273.

Fewster, R. (2014). Lectures Notes for Stats 325. https://www.stat.auckland.ac.nz/ fewster/325/notes/ch8.pdf.

Fielitz, B.D. and Bhargava, T.N. (1973). The behavior of stock price relatives: a Markovian analysis. Operations Research, **21**: 1183-1199.

Fielitz, B.D. (1975). On the stationarity of transition matrices of common stocks. Journal of Financial and Quantitative Analysis, **10**: 327-339.

D'Amico, G. and De Blasis, R. (2020). A multivariate Markov chain stock model, Scandinavian Actuarial Journal, **4**: 272-291. DOI: 10.1080/03461238.2019.1661280.

Hillier, F. S. and Lieberman, G. J. (1995). Introduction to Operations Research (6 ed.). McGraw-Hill, New York.

Jonsson, R. (2011). Tests of Markov order and homogeneity in a Markov chain model for analysing rehabilitation. Research Report, **7**, Statistical Research Unit, Department of Economics, Göteborg University, Sweden.

Mallikarjuna, M. and Rao, R.P. (2019). Evaluation of forecasting methods from selected stock market returns. Financ Innov., **5**(40). https://doi.org/10.1186/s40854-019-0157-x.

McQuenn, G. and Thorley, S. (1991). Are stock returns predictable? A test using Markov chains. Journal of Finance, **46**(1): 239-263.

Niederhoffer, V. and Osborne, M. (1966). Market making and reversal on the stock exchange. Journal of the American Statistical Association, **61**: 897-916.

Poterba, J.M. and Summers, L. H. (1988). Mean reversion in stock prices: evidence and implications. Journal of Financial Economics, **22**: 27-59.

Roberts, H. V. (1959). Stock market 'patterns' and financial analysis: methodological suggestions. The Journal of Finance, 14(1): 1-10.

Savills (2019). Savills report: How much is the world worth? https://www.smartowner.com/blog/savills-report-how-much-is-the-world-worth.

Shah, D., Isah, H. and Zulkernine, F. (2019). Stock Market Analysis: A Review and Taxonomy of Prediction Techniques. Int. J. Financial Stud., **7**, 26.

Sigman, C. (2009). Discrete-time Markov chains. http://www.columbia.edu/ ks20/stochastic-I/stochastic-I-MCI.pdf.

Singer, P. (2013). Order estimation for Markov models. $https : //medium.com/@ph_singer/order-estimation-for-markov-chain-models-6cde3ad2410b$.

Tan,B. and Yilmaz K. (2002). Markov chain test for time dependence and homogeneity: an analytical and empirical evaluation. European Journal of Operational Research, **137**(3): 524 – 543.

Zhong, X. and Enke, D. (2019). Predicting the daily return direction of the stock market using hybrid machine learning algorithms. Financial Innovation, **5**(4):1–20.

Appendix

Conditional Variance Dynamics

GARCH Model : sGARCH(1,1)
Mean Model : ARFIMA(1,0,1)
Distribution : std

Optimal Parameters:

	Estimate	Std. Error	t value	Pr(>|t|)
Mu	58.878675	3.371619	17.4630	0.000000
ar1	1.000000	0.001326	754.2313	0.000000
ma1	0.073342	0.032547	2.2534	0.024231
Omega	1.377778	0.473728	2.9084	0.003633
alpha1	0.351438	0.053284	6.5955	0.000000
beta1	0.647561	0.048795	13.2711	0.000000
Shape	4.028278	0.424686	9.4853	0.000000

Robust Standard Errors:

	Estimate	Std. Error	t value	Pr(>|t|)
Mu	58.878675	0.412657	142.6820	0.000000
ar1	1.000000	0.001940	515.5341	0.000000
ma1	0.073342	0.033511	2.1886	0.028628
Omega	1.377778	0.862366	1.5977	0.110116
alpha1	0.351438	0.053859	6.5251	0.000000
beta1	0.647561	0.071319	9.0797	0.000000
Shape	4.028278	0.339768	11.8560	0.000000

LogLikelihood : -2922.414

Information Criteria

```
Akaike        5.9061
Bayes         5.9407
Shibata       5.9060
Hannan-Quinn  5.9192
```

Weighted Ljung-Box Test on Standardized Residuals

```
                          statistic  p-value
Lag[1]                        1.296  2.55e-01
Lag[2*(p+q)+(p+q)-1][5]      10.588  1.31e-14
Lag[4*(p+q)+(p+q)-1][9]      15.076  3.48e-05
```

d.o.f=2

H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals

----

	statistic	p-value
Lag[1]	0.5726	0.4492
Lag[2*(p+q)+(p+q)-1][5]	2.3089	0.5474
Lag[4*(p+q)+(p+q)-1][9]	3.0946	0.7432

d.o.f=2

Weighted ARCH LM Tests

	Statistic	Shape	Scale	P-Value
ARCH Lag[3]	2.022	0.500	2.000	0.1551
ARCH Lag[5]	2.482	1.440	1.667	0.3742
ARCH Lag[7]	2.607	2.315	1.543	0.5905

Nyblom stability test

Joint Statistic: 7.073
Individual Statistics:

mu	0.0004783
ar1	1.4474682
ma1	0.3422050
omega	0.9946240
alpha1	1.5517017
beta1	0.8110123
shape	0.6790360

Asymptotic Critical
Values (10% 5% 1%)
Joint Statistic: 1.69
1.9 2.35
Individual Statistic: 0.35 0.47 0.75

Sign Bias Test

----

	t-value	prob sig
Sign Bias	0.57478	0.5656
Negative Sign Bias	0.03817	0.9696
Positive Sign Bias	1.18911	0.2347
Joint Effect	3.34597	0.3413

Adjusted Pearson Goodness-of-Fit Test:

----

	group	Statistic	p-value(g-1)
1	20	36.67	0.008725
2	30	54.61	0.002749
3	40	67.68	0.002966
4	50	73.73	0.012719

Elapsed time: 1.248002