

The Algebraic Structure of Eight-Run Minimum Aberration Two-Level Fractional Factorial Designs

V. U. Ekhosuehi*

Department of Statistics, University of Benin, P.M.B. 1154, Benin City, Nigeria

Abstract. This paper is concerned with the eight-run minimum aberration fractional factorial designs at two levels. The design is characterised by run size economy and aliasing effects. The paper aims to describe the features of the design from an algebraic perspective. It is found that the group formed by the aliasing sets of the design is a non-cyclic finite Abelian p -group. It is shown by examples that the partitioned aliasing subsets of the group, which are constructed according to the number of possible ways the identity relation can be obtained, are not subgroups and that a mapping defined on the maximum word length of an aliased set is not a homomorphism.

Keywords: defining relation, fractional factorial design, group, minimum aberration.

Published by: Department of Statistics, University of Benin, Nigeria

1. Introduction

Prior to the development of prediction equations involving important variables and the optimal process operating conditions of a system, a screening design is usually completed. The screening design, which is the initial stage of many industrial experiments and process development, is used to determine efficiently and effectively a small number of potential factors among a large number of factors that may affect a particular response (Georgiou et al., 2014; Alhelali et al., 2020). Traditional screening designs, such as the two-level fractional factorial designs, are commonly used to achieve this purpose after one set of experiment (Olsen et al., 2016). The designs are intended to find significant main effects, rather than interaction effects (cf. Montgomery and Jennings, 2006). This is unlike experiments based on full factorial designs, where all factors and their interactions are investigated (for instance, in Santamaria et al. (2019), a full 2^5 factorial design was conducted to quantify the effects of five construction site factors including crew experience, compaction method, mixing time, curing humidity and curing temperature).

Let 2^{k-q} be a regular fractional factorial design at two levels with k factors and 2^{k-q} runs constructed by choosing $q > 0$ independent fractional generators from the set of all factorial effects in the 2^k factorial layout. The integer q is the degree of fractionation. This kind of experiment is well known in the literature (Vincente et al., 1998; Wu and Hamada, 2000) and it has several economic benefits to the experimenter (Mee, 2009). The run size economy of the fractional factorial design results to aliasing effects. The aliasing effect is used to describe factorial effects whose estimates are indistinguishable. The q independent fractional generators may be chosen in different ways. As a consequence distinct sets of fractional generators may give distinct sets of designs. To select the 'best' design, certain criteria are used, for instance, the resolution and the minimum aberration (Mee, 2009; Wu and Hamada, 2000). Most researchers rely on the minimum aberration (MA) criterion as it can discriminate among designs of the same resolution. There are a number of papers in the literature, which discuss the concept and the practical use of the MA criterion in screening experiments. For details refer to Fries and Hunter (1980), Tang and Wu (1996), Chen and Hedayat (1998), Cheng and Mukerjee (1998), Hu and Zhang (2009) and Ekhosuehi et al. (2018).

Perhaps the MA 2^{k-q} fractional factorial designs are the most commonly used designs in screening experiments. The aliased sets from such designs together with the defining relation form an Abelian group

*Corresponding author. Email: virtue.ekhosuehi@uniben.edu

with multiplication being the corresponding column summation mod 2 (Chen, 1998; Liao and Chai, 2009). In spite of the statistical overtones on the successes that MA 2^{k-q} fractional factorial designs have had in screening experiments, it is important to know whether there are other algebraic properties of the design criterion. This study focuses on the theoretic aspects of the 2^{k-q} design. The applied aspects of the design can be found elsewhere (Montgomery and Jennings, 2006; Olsen et al., 2016). The theoretical underpinning of this study is hinged on the theory of finite Abelian groups. For a discussion on this theory and other algebraic statistics in design of experiments refer to Pistone et al. (2009).

This paper is written to assist in answering questions on MA 2^{k-q} fractional factorial designs of the following sort: (i) Is the group formed by the aliasing sets cyclic? (ii) Is it possible to find a subgroup from the partitioned subsets of the defining relation? (iii) Can we define a homomorphism using the word length? These questions on the structure of MA 2^{k-q} designs, which are used in screening experiments, are crucial in experimental design. As earlier mentioned in Bate and Chatfield (2016a), modern statistical software packages allow the non-statisticians to perform experimental design without correctly identifying the underlying structure of the experimental design and this can lead to incorrect model selection and misleading inferences. In another study, Bate and Chatfield (2016b) provided a practical situation where the structure of an experimental design coupled with randomisation was used in the construction of mixed model statistical analysis. Regardless of this practical problem, the algebraic structure of factorial designs sums up a problem in the literature (Bailey, 1985). Bailey (1985) had attempted this problem by using the theory of finite Abelian groups to simplify the search for and construction of factorial designs. However, the study restricted attention to designs with group homomorphism.

This study is aimed at describing the features of the eight-run MA 2^{k-q} fractional factorial designs from an algebraic perspective. It is important to mention here that several higher runs 2^{k-q} designs are available in the literature. For example, the screening experiment on the process of synthesising TiO_2 in the laboratory was carried out in Olsen et al. (2016) with a 16-run 2^{10-6}_{III} fractional factorial design. This study is restricted to the 8-run MA 2^{k-q} fractional factorial designs for convenience. Even so, the obtained results herein can be extended to other higher runs two-level fractional factorial designs.

To avoid confusion in the main part of this paper and make the paper clearer to a larger audience, the following definitions are given (Herstein, 1975).

DEFINITION 1.1 A nonempty set of elements G is said to form a group if in G there is defined a binary operation, called the product and denoted by $*$, such that

- i. $a, b \in G$ implies that $a * b \in G$ (closed)
- ii. $a, b, c \in G$ implies that $a * (b * c) = (a * b) * c$ (associative law)
- iii. There exists an element $e \in G$ such that $a * e = e * a = a$ for all $a \in G$ (the existence of an identity element in G)
- iv. For every $a \in G$, there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$ (the existence of inverses in G).

DEFINITION 1.2 A group G is said to be Abelian (or commutative) if for every $a, b \in G$, $a * b = b * a$.

DEFINITION 1.3 The order of a group G , denoted by $|G|$, is the number of elements it contains. If $|G| < \infty$, then G is said to be a finite group.

DEFINITION 1.4 If G is a group and $a \in G$, the order (or period) of a is the least positive integer m such that $a^m = e$. If no such integer exists, we say that a is of infinite order.

DEFINITION 1.5 Let G be a group and p a prime. A group of order p^n for some $n \geq 0$ is called a p -group.

DEFINITION 1.6 A mapping f from a group G into a group G' is called a homomorphism if, for every $a, b \in G$, $f(ab) = f(a)f(b)$. In addition, if f is one-to-one and onto, then f is called an isomorphism, and G and G' are said to be isomorphic.

Unless otherwise explained, all terminologies and notations used in this paper are consistent with that of Herstein (1975) and the experimental design literature (cf. Wu and Hamada, 2000).

2. Materials and method

This section contains the proof of the algebraic properties of the eight-run MA 2^{k-q} fractional factorial designs mentioned in the Introduction. In addition, a list of the members in the class of eight-run 2^{k-q} fractional factorials is provided and an expression for the binary product between any two aliased sets of the group in terms of the defining words is given.

Let $A = \{a_i : i = 1, 2, \dots, k\}$ represent the set of factor labels (or letters) for the k factors. We consider a special kind of set where the equality sign may as well be used to list the elements. The equality sign is used to indicate aliases, that is, the factor effects that cannot be estimated independently. Let $\mathbf{I}$ denote the identity element of the MA 2^{k-q} fractional factorial design. Then the defining relation of the form $\mathbf{I} = a_1 a_2 a_4 = a_1 a_3 a_5 = a_2 a_3 a_4 a_5$ is the set $\{\mathbf{I}, a_1 a_2 a_4, a_1 a_3 a_5, a_2 a_3 a_4 a_5\}$. The elements of the set of defining relation are called words and the number of factors in each word is the word length. The aliasing sets are derived from the defining relation. In this paper, the notation G_u , where u is a positive integer, is used to denote the aliasing sets.

Let $B_j = \left\{ a_j : a_j = \prod_{r \neq j} a_r, \quad a_r \in A \right\}$ be the defining word of a defining contrast subgroup of word length $m_j = \left| \prod_{r \neq j} a_r \right|$, $j = k-q+1, k-q+2, \dots, k$. The set $\bigcup_{j=k-q+1}^k B_j$, such that $\bigcap_{j=k-q+1}^k B_j = \emptyset$, is called the set of design generators. The number of words in the set of design generators is q . For the q defining words, the total number of identity relations is expressed as $\sum_{r=1}^q \binom{q}{r}$. The product of a defining word with itself gives an identity, $\mathbf{I}$, that is, $\mathbf{I} = a_j \prod_{r \neq j} a_r, \quad a_r \in A$. Equating these identities gives the defining relation. The defining relation is also called the defining contrast subgroup. The defining contrast subgroup may be partitioned according to the number of possible ways the identity relation can be obtained. Let S_{k-q} be the defining set, that is, the set of all possible combinations of the defining words in $\bigcup_{j=k-q+1}^k B_j$ that would give the identity relation, $\mathbf{I}$, for the fractional factorial with $k-q$ main factors. This set, S_{k-q} , uniquely determines a regular fractional factorial design.

Let d denote the regular 2^{k-q} fractional factorial design. From the defining set, S_{k-q} , the word length pattern of the design is determined. The word length pattern of the design, d , is a vector $W(d) = (A_3(d), A_4(d), \dots, A_k(d))$, where $A_r(d)$ is the number of words of length r in S_{k-q} with $\sum_{r=3}^k A_r(d) = \sum_{r=1}^q \binom{q}{r}$ (Chen, 1998). The summand $\binom{q}{r} = \frac{q!}{r!(q-r)!}$ corresponds to the subtotal of the words of length r in S_{k-q} . The resolution of the design d is the smallest r satisfying $A_r(d) \geq 1$. The design d is said to have minimum aberration if it has the minimum number of words in S_{k-q} among other designs (Fries and Hunter, 1980).

Let G be a group formed by the aliasing sets, G_u , together with the identity element, represented as G_0 , of the MA 2^{k-q} fractional factorial design. That is, the aliasing sets, G_u , and the identity element, G_0 , are elements of G . The identity element, G_0 , is the defining relation from which the other elements of G are derived. Let $\varphi(G_u, G_v)$ be the number of letters common to the maximum length of the defining words of the aliased sets $G_u, G_v \in G$, and such that $\varphi(G_u, G_u) = m(G_u)$ and $\varphi(G_u, G_0) = 0$. The notation $m(G_u)$ denotes the maximum length of the defining word of the aliased set G_u . Using these notations and symbols, the following propositions are stated with their proofs.

PROPOSITION 2.1 *The group G formed by the aliasing sets together with the identity relation of the MA 2^{k-q} fractional factorial design is a non-cyclic, finite Abelian p -group.*

Proof. For every MA 2^{k-q} fractional factorial design, there are $2^{k-q} - 1$ columns generated by the $k - q$ independent columns. These $2^{k-q} - 1$ columns comprise of factorial effects that are aliased. The $2^{k-q} - 1$ aliased sets together with the defining relation $\mathbf{I}$ form a group G and the order of the group is $|G| = 2^{k-q}$. Since 2 is a prime number with $k - q > 0$, the group G is a p -group. Moreover, $2^{k-q} < \infty$ implies that G is a finite group. For each element $G_u \in G$, $G_u * G_u = \mathbf{I}$ implies that $|G_u| = 2$ and that the generating set of G_u is $\{\mathbf{I}, G_u\}$. Since every element of G has order 2, none of the elements of G generates G . Thus, G is not cyclic. For all $G_u, G_v \in G$, $G_u \neq G_v$, $G_u * G_v \in G$ (by the property of closure). Since every element of G has order 2, $(G_u * G_v) * (G_u * G_v) = \mathbf{I}$. Multiplying from the left hand side by G_u and from the right hand side by G_v , $G_u * (G_u * G_v) * (G_u * G_v) * G_v = G_u * G_v$. By the property of associativity, the operation is the same as $(G_u * G_u) * G_v * G_u * (G_v * G_v) = G_u * G_v$, which simplifies to $G_v * G_u = G_u * G_v$. Hence, G is Abelian. ■

PROPOSITION 2.2 Let $Z_{k+1} = \{0, 1, 2, \dots, k\}$ be a set of integer modulo $k + 1$. The mapping $m : G \rightarrow Z_{k+1}$, where m is the maximum length of the defining word of an aliased set in G , is not a homomorphism, and hence G and Z_{k+1} are not isomorphic.

Proof. Consider the aliased sets $G_u, G_v \in G$, $u \neq v$. Suppose we count the words of the maximum length of the defining words in G_u and then count the one in G_v . Then every letter common to both G_u and G_v would have been counted twice. Since $G_u * G_v$ is another element in G with all the letters either in G_u or G_v , but not both (as $a_j * a_j = \mathbf{I}$, $a_j \in A$), then every letter common to the maximum length of the defining words of the aliased sets G_u and G_v should be deleted in $G_u * G_v$. Hence, the binary product of any two aliased sets $G_u, G_v \in G$, $u \neq v$, in terms of the function m is given as

$$m(G_u * G_v) = m(G_u) + m(G_v) - 2\varphi(G_u, G_v) \neq m(G_u)m(G_v).$$

Moreover, for $u = v$, $G_u * G_u = G_0$ so that

$$m(G_0) = m(G_u * G_u) = m(G_u) + m(G_u) - 2\varphi(G_u, G_u) = 0$$

as $\varphi(G_u, G_u) = m(G_u)$. For $G_u \neq G_0$, $m(G_u * G_u) \neq m(G_u)m(G_u)$. Similarly, $m(G_u * G_0) \neq m(G_u)m(G_0)$. Hence m is not a homomorphism. Since m is not a homomorphism, then G and Z_{k+1} are not isomorphic. ■

PROPOSITION 2.3 For the 2^{k-q} fractional factorial designs with $k - q = 3$, $k \leq 7$ and $0 < q \leq 4$.

Proof. For a fractional factorial design to exist, the degree of fractionation q must be a positive integer. So $q > 0$. For the $k - q$ main factors, the number of possible interactions may be 2, 3, $\dots$, or $k - q$, so that the possible number of ways in which the design generators can be constructed are $\binom{k-q}{2}$, $\binom{k-q}{3}$, $\dots$ or $\binom{k-q}{k-q}$. The total number of possible ways is $\sum_{r=3}^{k-q} \binom{k-q}{r-1}$, which is an upper bound for the degree of fractionation, q , that is,

$$q \leq \sum_{r=3}^{k-q} \binom{k-q}{r-1}.$$

With $k - q = 3$, $q \leq 4$. Hence, $k \leq 7$ and $0 < q \leq 4$. ■

3. Illustrative examples

In this section it is shown by examples that the partitioned aliasing subsets of the MA 2^{k-q} design with $k - q = 3$, which are constructed according to the number of possible ways the identity relation can be obtained, are not subgroups. It is convenient to describe the MA 2^{k-q} designs with $k - q = 3$ by considering the multiplication tables for the designs. From Proposition 2.3, the list of members in the class of eight-run regular 2^{k-q} design are 2^{4-1} , 2^{5-2} , 2^{6-3} and 2^{7-4} . Using the MA criterion, it can be verified that the MA 2^{k-q} designs with $k - q = 3$ are:

- (1) The MA 2_{IV}^{4-1} design with $B_4 = \{a_4 : a_4 = a_1 a_2 a_3\}$,
 - (2) The MA 2_{III}^{5-2} design with $B_4 = \{a_4 : a_4 = a_1 a_2\}$, $B_5 = \{a_5 : a_5 = a_1 a_3\}$,
 - (3) The MA 2_{III}^{6-3} design with $B_4 = \{a_4 : a_4 = a_1 a_2\}$, $B_5 = \{a_5 : a_5 = a_1 a_3\}$, $B_6 = \{a_6 : a_6 = a_2 a_3\}$,
 - (4) The MA 2_{III}^{7-4} design with $B_4 = \{a_4 : a_4 = a_1 a_2\}$, $B_5 = \{a_5 : a_5 = a_1 a_3\}$, $B_6 = \{a_6 : a_6 = a_2 a_3\}$, $B_7 = \{a_7 : a_7 = a_1 a_2 a_3\}$,
- where the subscript in the MA 2^{k-q} fractional factorials denotes the resolution of the design.

Let $H_i^{2^{k-q}} \subseteq G$ be the partitioned aliasing sets of an MA 2^{k-q} design with $k - q = 3$ and resolution R for the decomposition index i . This study examines the binary operation between any two elements in $H_i^{2^{k-q}}$

for each class of MA 2^{k-q} design with $k - q = 3$ and constructs multiplication tables for the outcomes. The idea behind the multiplication is based on ensuring that all the factor interactions in $G_u * G_v$ are contained in $H_i^{2^{k-q}}$.

EXAMPLE 3.1 MA 2_{IV}^{4-1} design with $B_4 = \{a_4 : a_4 = a_1 a_2 a_3\}$

The defining set for the MA 2_{IV}^{4-1} design is given as

$$S_{4-1} = \{\mathbf{I} = a_1 a_2 a_3 a_4\}.$$

Apparently, the defining relation, $\mathbf{I}$, cannot be partitioned into subsets. The set

$$H_1^{2_{IV}^{4-1}} = \{G_0, G_1, G_2, G_3, G_4, G_{12}, G_{13}, G_{14}\} = G,$$

where the members of $H_1^{2_{IV}^{4-1}}$ are given as follows. $G_0 : \mathbf{I} = a_1 a_2 a_3 a_4$, $G_1 : a_1 = a_2 a_3 a_4$, $G_2 : a_2 = a_1 a_3 a_4$, $G_3 : a_3 = a_1 a_2 a_4$, $G_4 : a_4 = a_1 a_2 a_3$, $G_{12} : a_1 a_2 = a_3 a_4$, $G_{13} : a_1 a_3 = a_2 a_4$ and $G_{14} : a_1 a_4 = a_2 a_3$.

Without loss of generality we omit the curly braces from the aliasing sets. The multiplication table for the MA 2_{IV}^{4-1} design is shown in Table 1.

Table 1: Multiplication table for the MA 2_{IV}^{4-1} design

*	G_0	G_1	G_2	G_3	G_4	G_{12}	G_{13}	G_{14}
G_0	G_0	G_1	G_2	G_3	G_4	G_{12}	G_{13}	G_{14}
G_1	G_1	G_0	G_{12}	G_{13}	G_{14}	G_2	G_3	G_4
G_2	G_2	G_{12}	G_0	G_{14}	G_{13}	G_1	G_4	G_3
G_3	G_3	G_{13}	G_{14}	G_0	G_{12}	G_4	G_1	G_2
G_4	G_4	G_{14}	G_{13}	G_{12}	G_0	G_3	G_2	G_1
G_{12}	G_{12}	G_2	G_1	G_4	G_3	G_0	G_{14}	G_{13}
G_{13}	G_{13}	G_3	G_4	G_1	G_2	G_{14}	G_0	G_{12}
G_{14}	G_{14}	G_4	G_3	G_2	G_1	G_{13}	G_{12}	G_0

In Example 3.2 and the subsequent ones, A_r is used to denote the r -th subset of the defining set S_{k-q} . The subset A_r is constructed in such fashion that the intersection $A_i \cap A_j = \emptyset$ for $i \neq j$. In these subsequent examples, it is possible to partition the defining sets. The sets are partitioned according to the number of possible ways the identity relation can be obtained. The added aliased sets are generated to include the main effects and factor interactions that would complete the multiplication table. The dash in the resulting multiplication table is used to indicate operations which are undefined.

EXAMPLE 3.2 MA 2_{III}^{5-2} design with $B_4 = \{a_4 : a_4 = a_1 a_2\}$, $B_5 = \{a_5 : a_5 = a_1 a_3\}$

The defining set for the MA 2_{III}^{5-2} design is obtained as

$$S_{5-2} = \{\mathbf{I} = a_1 a_2 a_4 = a_1 a_3 a_5 = a_2 a_3 a_4 a_5\}.$$

The defining set, S_{5-2} , is partitioned into two parts. The results emanating from the partitioning alongside with the multiplication tables are presented as follows.

A. Let $A_1 = \{\mathbf{I} = a_1 a_2 a_4 = a_1 a_3 a_5\}$. Using the subset $A_1 \subset S_{5-2}$, we get

$$H_1^{2_{III}^{5-2}} = \{G_0, G_1, G_2, G_3, G_4, G_5, G_{23}, G_{25}\} \subset G,$$

where $G_0 : \mathbf{I} = a_1 a_2 a_4 = a_1 a_3 a_5$, $G_1 : a_1 = a_2 a_4 = a_3 a_5$, $G_2 : a_2 = a_1 a_4 = a_1 a_3 a_4 a_5$, $G_3 : a_3 = a_1 a_2 a_3 a_4 = a_1 a_5$, $G_4 : a_4 = a_1 a_2 = a_1 a_3 a_4 a_5$, $G_5 : a_5 = a_1 a_2 a_4 a_5 = a_1 a_3$, $G_{23} : a_2 a_3 = a_1 a_3 a_4 = a_1 a_2 a_5$ and $G_{25} : a_2 a_5 = a_1 a_4 a_5 = a_1 a_2 a_3$.

The multiplication table for $H_1^{2_{III}^{5-2}}$ is given in Table 2.

Table 2: Multiplication table for the MA 2_{III}^{5-2} design with $\mathbf{I} = a_1a_2a_4 = a_1a_3a_5$

*	G_0	G_1	G_2	G_3	G_4	G_5	G_{23}	G_{25}
G_0	G_0	G_1	G_2	G_3	G_4	G_5	G_{23}	G_{25}
G_1	G_1	G_0	G_4	G_5	G_2	G_3	—	—
G_2	G_2	G_4	G_0	G_{23}	G_1	G_{25}	G_3	G_5
G_3	G_3	G_5	G_{23}	G_0	—	G_1	—	—
G_4	G_4	G_2	G_1	—	G_0	—	—	—
G_5	G_5	G_3	G_{25}	G_1	—	G_0	—	—
G_{23}	G_{23}	—	G_3	—	—	—	G_0	—
G_{25}	G_{25}	—	G_5	—	—	—	—	G_0

B. With $A_2 = \{\mathbf{I} = a_2a_3a_4a_5\}$, we get

$$H_2^{2^{5-2}} = \{G_0, G_1, G_2, G_3, G_4, G_5, G_{34}, G_{45}\} \subset G,$$

where $G_0 : \mathbf{I} = a_2a_3a_4a_5$, $G_1 : a_1 = a_1a_2a_3a_4a_5$, $G_2 : a_2 = a_3a_4a_5$, $G_3 : a_3 = a_2a_4a_5$, $G_4 : a_4 = a_2a_3a_5$, $G_5 : a_5 = a_2a_3a_4$, $G_{34} : a_3a_4 = a_2a_5$ and $G_{45} : a_4a_5 = a_2a_3$.

Table 3 shows the multiplication that exists in the subset.

Table 3: Multiplication table for the MA 2_{III}^{5-2} design with $\mathbf{I} = a_2a_3a_4a_5$

*	G_0	G_1	G_2	G_3	G_4	G_5	G_{34}	G_{45}
G_0	G_0	G_1	G_2	G_3	G_4	G_5	G_{34}	G_{45}
G_1	G_1	G_0	—	—	—	—	—	—
G_2	G_2	—	G_0	G_{45}	—	G_{34}	G_5	G_3
G_3	G_3	—	G_{45}	G_0	G_{34}	—	G_4	G_2
G_4	G_4	—	—	G_{34}	G_0	G_{45}	G_3	G_5
G_5	G_5	—	G_{34}	—	G_{45}	G_0	G_2	G_4
G_{34}	G_{34}	—	G_5	G_4	G_3	G_2	G_0	—
G_{45}	G_{45}	—	G_3	G_2	G_5	G_4	—	G_0

EXAMPLE 3.3 MA 2_{III}^{6-3} design with $B_4 = \{a_4 : a_4 = a_1a_2\}$, $B_5 = \{a_5 : a_5 = a_1a_3\}$, $B_6 = \{a_6 : a_6 = a_2a_3\}$

In this case the defining set is

$$S_{6-3} = \{\mathbf{I} = a_1a_2a_4 = a_1a_3a_5 = a_2a_3a_6 = a_2a_3a_4a_5 = a_1a_3a_4a_6 = a_1a_2a_5a_6 = a_4a_5a_6\}.$$

The defining relation, $\mathbf{I}$, can be partitioned into three subsets, A_1, A_2, A_3 as shown below together with its corresponding multiplication tables.

A. With $A_1 = \{\mathbf{I} = a_1a_2a_4 = a_1a_3a_5 = a_2a_3a_6\}$, we obtain

$$H_1^{2^{6-3}} = \{G_0, G_1, G_2, G_3, G_4, G_5, G_6, G_{16}\} \subset G,$$

where $G_0 : \mathbf{I} = a_1a_2a_4 = a_1a_3a_5 = a_2a_3a_6$, $G_1 : a_1 = a_2a_4 = a_3a_5 = a_1a_2a_3a_6$, $G_2 : a_2 = a_1a_4 = a_1a_2a_3a_5 = a_3a_6$, $G_3 : a_3 = a_1a_2a_3a_4 = a_1a_5 = a_2a_6$, $G_4 : a_4 = a_1a_2 = a_1a_3a_4a_5 = a_2a_3a_4a_6$, $G_5 : a_5 = a_1a_2a_4a_5 = a_1a_3 = a_2a_3a_5a_6$, $G_6 : a_6 = a_1a_2a_4a_6 = a_1a_3a_5a_6 = a_2a_3$ and $G_{16} : a_1a_6 = a_2a_4a_6 = a_3a_5a_6 = a_1a_2a_3$.

The multiplication is given in Table 4.

B. With $A_2 = \{\mathbf{I} = a_2a_3a_4a_5 = a_1a_3a_4a_6 = a_1a_2a_5a_6\}$, we have

$$H_2^{2^{6-3}} = \{G_0, G_1, G_2, G_3, G_4, G_5, G_6, G_{25}\} \subset G,$$

Table 4: Multiplication table for the MA 2_{III}^{6-3} design with $\mathbf{I} = a_1a_2a_4 = a_1a_3a_5 = a_2a_3a_6$

*	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_{16}
G_0	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_{16}
G_1	G_1	G_0	G_4	G_5	G_2	G_3	G_{16}	G_6
G_2	G_2	G_4	G_0	G_6	G_1	—	G_3	—
G_3	G_3	G_5	G_6	G_0	—	G_1	G_2	—
G_4	G_4	G_2	G_1	—	G_0	—	—	—
G_5	G_5	G_3	—	G_1	—	G_0	—	—
G_6	G_6	G_{16}	G_3	G_2	—	—	G_0	G_1
G_{16}	G_{16}	G_6	—	—	—	—	G_1	G_0

where $G_0 : \mathbf{I} = a_2a_3a_4a_5 = a_1a_3a_4a_6 = a_1a_2a_5a_6$, $G_1 : a_1 = a_1a_2a_3a_4a_5 = a_3a_4a_6 = a_2a_5a_6$, $G_2 : a_2 = a_3a_4a_5 = a_1a_2a_3a_4a_6 = a_1a_5a_6$, $G_3 : a_3 = a_2a_4a_5 = a_1a_4a_6 = a_1a_2a_3a_5a_6$, $G_4 : a_4 = a_2a_3a_5 = a_1a_3a_6 = a_1a_2a_4a_5a_6$, $G_5 : a_5 = a_2a_3a_4 = a_1a_3a_4a_5a_6 = a_1a_2a_6$, $G_6 : a_6 = a_2a_3a_4a_5a_6 = a_1a_3a_4 = a_1a_2a_5$ and $G_{25} : a_2a_5 = a_3a_4 = a_1a_2a_3a_4a_5a_6 = a_1a_6$.

Table 5 shows the results of the operation $G_u * G_v \in H_2^{2_{III}^{6-3}}$.

Table 5: Multiplication table for the MA 2_{III}^{6-3} design with $\mathbf{I} = a_2a_3a_4a_5 = a_1a_3a_4a_6 = a_1a_2a_5a_6$

*	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_{25}
G_0	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_{25}
G_1	G_1	G_0	—	—	—	—	G_{25}	G_6
G_2	G_2	—	G_0	—	—	G_{25}	—	G_5
G_3	G_3	—	—	G_0	G_{25}	—	—	G_4
G_4	G_4	—	—	G_{25}	G_0	—	—	G_3
G_5	G_5	—	G_{25}	—	—	G_0	—	G_2
G_6	G_6	G_{25}	—	—	—	—	G_0	G_1
G_{25}	G_{25}	G_6	G_5	G_4	G_3	G_2	G_1	G_0

C. With $A_3 = \{\mathbf{I} = a_4a_5a_6\}$, we get

$$H_3^{2_{III}^{6-3}} = \{G_0, G_1, G_2, G_3, G_4, G_5, G_6, G_{14}\} \subset G,$$

where $G_0 : \mathbf{I} = a_4a_5a_6$, $G_1 : a_1 = a_1a_4a_5a_6$, $G_2 : a_2 = a_2a_4a_5a_6$, $G_3 : a_3 = a_3a_4a_5a_6$, $G_4 : a_4 = a_5a_6$, $G_5 : a_5 = a_4a_6$, $G_6 : a_6 = a_4a_5$ and $G_{25} : a_1a_4 = a_1a_5a_6$.

The results of the multiplication is given in Table 6.

Table 6: Multiplication table for the MA 2_{III}^{6-3} design with $\mathbf{I} = a_4a_5a_6$

*	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_{14}
G_0	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_{14}
G_1	G_1	G_0	—	—	G_{14}	—	—	—
G_2	G_2	—	G_0	—	—	—	—	—
G_3	G_3	—	—	G_0	—	—	—	—
G_4	G_4	G_{14}	—	—	G_0	G_6	G_5	—
G_5	G_5	—	—	—	G_6	G_0	G_4	—
G_6	G_6	—	—	—	G_5	G_4	G_0	—
G_{14}	G_{14}	—	—	—	—	—	—	G_0

EXAMPLE 3.4 MA 2_{III}^{7-4} design with $B_4 = \{a_4 : a_4 = a_1a_2\}$, $B_5 = \{a_5 : a_5 = a_1a_3\}$, $B_6 = \{a_6 : a_6 = a_2a_3\}$, $B_7 = \{a_7 : a_7 = a_1a_2a_3\}$

The defining set for this design is

$$S_{7-4} = \{\mathbf{I} = a_1a_2a_4 = a_1a_3a_5 = a_2a_3a_6 = a_1a_2a_3a_7 = a_2a_3a_4a_5 = a_1a_3a_4a_6 = a_3a_4a_7 = a_1a_2a_5a_6 = a_2a_5a_7 = a_1a_6a_7 = a_4a_5a_6 = a_1a_4a_5a_7 = a_2a_4a_6a_7 = a_3a_5a_6a_7 = a_1a_2a_3a_4a_5a_6a_7\}.$$

The defining relation, $\mathbf{I}$, can be partitioned into subsets as follows.

A. With $A_1 = \{\mathbf{I} = a_1a_2a_4 = a_1a_3a_5 = a_2a_3a_6 = a_1a_2a_3a_7\}$, we have

$$H_1^{2_{III}^{7-4}} = \{G_0, G_1, G_2, G_3, G_4, G_5, G_6, G_7\} \subset G,$$

where $G_0 : \mathbf{I} = a_1a_2a_4 = a_1a_3a_5 = a_2a_3a_6 = a_1a_2a_3a_7$, $G_1 : a_1 = a_2a_4 = a_3a_5 = a_1a_2a_3a_6 = a_2a_3a_7$, $G_2 : a_2 = a_1a_4 = a_1a_2a_3a_5 = a_3a_6 = a_1a_3a_7$, $G_3 : a_3 = a_1a_2a_3a_4 = a_1a_5 = a_2a_6 = a_1a_2a_7$, $G_4 : a_4 = a_1a_2 = a_1a_3a_4a_5 = a_2a_3a_4a_6 = a_1a_2a_3a_4a_7$, $G_5 : a_5 = a_1a_2a_4a_5 = a_1a_3 = a_2a_3a_5a_6 = a_1a_2a_3a_5a_7$, $G_6 : a_6 = a_1a_2a_4a_6 = a_1a_3a_5a_6 = a_2a_3 = a_1a_2a_3a_6a_7$ and $G_7 : a_7 = a_1a_2a_4a_7 = a_1a_3a_5a_7 = a_2a_3a_6a_7 = a_1a_2a_3$.

The results of the binary operation $G_u * G_v \in H_1^{2_{III}^{7-4}}$ are presented in Table 7.

Table 7: Multiplication table for the MA 2_{III}^{7-4} design with $\mathbf{I} = a_1a_2a_4 = a_1a_3a_5 = a_2a_3a_6 = a_1a_2a_3a_7$

*	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_0	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_1	G_1	G_0	G_4	G_5	G_2	G_3	—	—
G_2	G_2	G_4	G_0	G_6	G_1	—	G_3	—
G_3	G_3	G_5	G_6	G_0	—	G_1	G_2	—
G_4	G_4	G_2	G_1	—	G_0	—	—	—
G_5	G_5	G_3	—	G_1	—	G_0	—	—
G_6	G_6	—	G_3	G_2	—	—	G_0	—
G_7	G_7	—	—	—	—	—	—	G_0

B. With $A_2 = \{\mathbf{I} = a_2a_3a_4a_5 = a_1a_3a_4a_6 = a_3a_4a_7 = a_1a_2a_5a_6 = a_2a_5a_7 = a_1a_6a_7\}$, we get

$$H_2^{2_{III}^{7-4}} = \{G_0, G_1, G_2, G_3, G_4, G_5, G_6, G_7\} \subset G,$$

where $G_0 : \mathbf{I} = a_2a_3a_4a_5 = a_1a_3a_4a_6 = a_3a_4a_7 = a_1a_2a_5a_6 = a_2a_5a_7 = a_1a_6a_7$, $G_1 : a_1 = a_1a_2a_3a_4a_5 = a_3a_4a_6 = a_1a_3a_4a_7 = a_2a_5a_6 = a_1a_2a_5a_7 = a_6a_7$, $G_2 : a_2 = a_3a_4a_5 = a_1a_2a_3a_4a_6 = a_2a_3a_4a_7 = a_1a_5a_6 = a_5a_7 = a_1a_2a_6a_7$, $G_3 : a_3 = a_2a_4a_5 = a_1a_4a_6 = a_4a_7 = a_1a_2a_3a_5a_6 = a_2a_3a_5a_7 = a_1a_3a_6a_7$, $G_4 : a_4 = a_2a_3a_5 = a_1a_3a_6 = a_3a_7 = a_1a_2a_4a_5a_6 = a_2a_4a_5a_7 = a_1a_4a_6a_7$, $G_5 : a_5 = a_2a_3a_4 = a_1a_3a_4a_5a_6 = a_3a_4a_5a_7 = a_1a_2a_6 = a_2a_7 = a_1a_5a_6a_7$, $G_6 : a_6 = a_2a_3a_4a_5a_6 = a_1a_3a_4 = a_3a_4a_6a_7 = a_1a_2a_5 = a_2a_5a_6a_7 = a_1a_7$ and $G_7 : a_7 = a_2a_3a_4a_5a_7 = a_1a_3a_4a_6a_7 = a_3a_4 = a_1a_2a_5a_6a_7 = a_2a_5 = a_1a_6$.

The multiplication results are presented in Table 8.

C. With $A_3 = \{\mathbf{I} = a_4a_5a_6 = a_1a_4a_5a_7 = a_2a_4a_6a_7 = a_3a_5a_6a_7\}$, we obtain

$$H_3^{2_{III}^{7-4}} = \{G_0, G_1, G_2, G_3, G_4, G_5, G_6, G_7\} \subset G,$$

where $G_0 : \mathbf{I} = a_4a_5a_6 = a_1a_4a_5a_7 = a_2a_4a_6a_7 = a_3a_5a_6a_7$, $G_1 : a_1 = a_1a_4a_5a_6 = a_4a_5a_7 = a_1a_2a_4a_6a_7 = a_1a_3a_5a_6a_7$, $G_2 : a_2 = a_2a_4a_5a_6 = a_1a_2a_4a_5a_7 = a_4a_6a_7 = a_2a_3a_5a_6a_7$, $G_3 : a_3 = a_3a_4a_5a_6 = a_1a_3a_4a_5a_7 = a_2a_3a_4a_6a_7 = a_5a_6a_7$, $G_4 : a_4 = a_5a_6 = a_1a_5a_7 = a_2a_6a_7 = a_3a_4a_5a_6a_7$, $G_5 : a_5 = a_4a_6 = a_1a_4a_7 = a_2a_4a_5a_6a_7 = a_3a_6a_7$, $G_6 : a_6 = a_4a_5 = a_1a_4a_5a_6a_7 = a_2a_4a_7 = a_3a_5a_7$ and $G_7 : a_7 = a_4a_5a_6a_7 = a_1a_4a_5 = a_2a_4a_6 = a_3a_5a_6$.

Table 9 contains the results of the binary operation $G_u * G_v \in H_3^{2_{III}^{7-4}}$.

D. With $A_4 = \{\mathbf{I} = a_1a_2a_3a_4a_5a_6a_7\}$, we get

$$H_4^{2_{III}^{7-4}} = \{G_0, G_1, G_2, G_3, G_4, G_5, G_6, G_7\} \subset G,$$

Table 8: Multiplication table for the MA 2_{III}^{7-4} design with $\mathbf{I} = a_2a_3a_4a_5 = a_1a_3a_4a_6 = a_3a_4a_7 = a_1a_2a_5a_6 = a_2a_5a_7 = a_1a_6a_7$

*	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_0	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_1	G_1	G_0	—	—	—	—	G_7	G_6
G_2	G_2	—	G_0	—	—	G_7	—	G_5
G_3	G_3	—	—	G_0	G_7	—	—	G_4
G_4	G_4	—	—	G_7	G_0	—	—	G_3
G_5	G_5	—	G_7	—	—	G_0	—	G_2
G_6	G_6	G_7	—	—	—	—	G_0	G_1
G_7	G_7	G_6	G_5	G_4	G_3	G_2	G_1	G_0

Table 9: Multiplication table for the MA 2_{III}^{7-4} design with $\mathbf{I} = a_4a_5a_6 = a_1a_4a_5a_7 = a_2a_4a_6a_7 = a_3a_5a_6a_7$

*	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_0	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_1	G_1	G_0	—	—	—	—	—	—
G_2	G_2	—	G_0	—	—	—	—	—
G_3	G_3	—	—	G_0	—	—	—	—
G_4	G_4	—	—	—	G_0	G_6	G_5	—
G_5	G_5	—	—	—	G_6	G_0	G_4	—
G_6	G_6	—	—	—	G_5	G_4	G_0	—
G_7	G_7	—	—	—	—	—	—	G_0

where $G_0 : \mathbf{I} = a_1a_2a_3a_4a_5a_6a_7$, $G_1 : a_1 = a_2a_3a_4a_5a_6a_7$, $G_2 : a_2 = a_1a_3a_4a_5a_6a_7$, $G_3 : a_3 = a_1a_2a_4a_5a_6a_7$, $G_4 : a_4 = a_1a_2a_3a_5a_6a_7$, $G_5 : a_5 = a_1a_2a_3a_4a_6a_7$, $G_6 : a_6 = a_1a_2a_3a_4a_5a_7$ and $G_7 : a_7 = a_1a_2a_3a_4a_5a_6$.

Table 10 includes the results of the binary operation $G_u * G_v \in H_4^{2_{III}^{7-4}}$ for this subset.

Table 10: Multiplication table for the MA 2_{III}^{7-4} design with $\mathbf{I} = a_1a_2a_3a_4a_5a_6a_7$

*	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_0	G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_1	G_1	G_0	—	—	—	—	—	—
G_2	G_2	—	G_0	—	—	—	—	—
G_3	G_3	—	—	G_0	—	—	—	—
G_4	G_4	—	—	—	G_0	—	—	—
G_5	G_5	—	—	—	—	G_0	—	—
G_6	G_6	—	—	—	—	—	G_0	—
G_7	G_7	—	—	—	—	—	—	G_0

The results of the analysis of the partitioned aliasing subsets of the MA 2^{k-q} design with $k - q = 3$ have led to several conclusions. It is clear that with $q = 1$, the defining relation cannot be partitioned into any other aliasing subset and that the multiplication table for $q = 1$ is that of the Abelian group, G . Suppose that $q > 1$. Then it is sufficient to show that the property of closure does not hold for the aliasing subsets G_u in $H_i^{2_R^{k-q}}$, which is derived from A_i . From the multiplication tables (Table 2 – 10) for the aliasing subsets G_u in $H_i^{2_R^{k-q}}$ that have been generated, it appears that some of the binary operations $G_u * G_v$ are undefined in $H_i^{2_R^{k-q}}$. This is so because some of the elements $G_u * G_v$ do not give an equivalent aliasing subset in $H_i^{2_R^{k-q}}$. As shown in Table 2 with $S_{k-q} \setminus \{a_2a_3a_4a_5\}$ as the identity relation for the MA 2_{III}^{5-2} design, $G_3, G_4 \in H_1^{2_{III}^{5-2}}$, but $G_3 * G_4 \notin H_1^{2_{III}^{5-2}}$. Once more, for $G_1, G_{23} \in H_1^{2_{III}^{5-2}}$, $G_3 * G_4 : a_1a_2a_3 = a_3a_4 = a_2a_5$. There is no aliased subsets in $H_1^{2_{III}^{5-2}}$ which contains a_3a_4 ; hence $G_1 * G_{23} \notin H_1^{2_{III}^{5-2}}$. However, the union $H_1^{2_{III}^{5-2}} \cup H_2^{2_{III}^{5-2}}$ of

the MA 2_{III}^{5-2} design forms an Abelian group, which is G . Similar results hold for MA 2_{III}^{6-3} and MA 2_{III}^{7-4} , respectively.

The multiplication tables also show that the union $\bigcup_{i=1, i \neq l}^q H_i^{2^{k-q}}$ with $q > 1$ for some l does not form a subgroup for the MA 2_R^{k-q} designs, whenever each of the subsets in $H_i^{2^{k-q}}$ contains the aliased subsets, G_u , $u \geq 1$, wherein a single factor or two factor interactions are aliased with other two factor interactions. However, $\bigcup_{i=1, i \neq 4}^4 H_i^{2_{III}^{7-4}}$ is a subgroup for the MA 2_{III}^{7-4} design because the aliased subsets in $H_4^{2_{III}^{7-4}}$ contains the aliased subsets G_u , where all the single factors are aliased with the six factor interactions. Hence, the partitioned aliasing subsets of the group, which are constructed according to the number of possible ways the identity relation of the MA 2_R^{k-q} designs with $k - q = 3$ can be obtained, are not subgroups.

4. Conclusion

The work presented in this paper has been centred on the algebraic aspects of the eight-run fractional factorial designs constructed using design generators that define q added columns in terms of interactions of the $k - q$ basic columns, where the factorial effect columns are indistinguishable from their aliases. It is demonstrated that the aliasing sets together with the defining relation of the eight-run MA 2^{k-q} fractional factorial designs have the following algebraic properties:

- It is a finite Abelian (or commutative) p -group with $p = 2$.
- It is not cyclic.
- The partitioned sets, which are constructed according to the ways the identity relation can be obtained, are not subgroups.
- The word length function m of an aliasing set is not a homomorphism.
- The group formed by the aliasing sets and the set of integer modulo $k + 1$ based on the mapping $m : G \rightarrow Z_{k+1}$ are not isomorphic.

It is worthwhile to extend the obtained results in this study to other regular MA 2^{k-q} fractional factorials with higher runs and to verify whether similar results hold for non-regular 2^{k-q} factorial designs.

Acknowledgement

The author is grateful to the anonymous referees for their valuable suggestions the incorporation of which has enhanced the presentation and ideas of the present paper.

References

- Alhelali, M. H., Georgiou, S. D. and Stylianou, S. (2020). Screening designs based on weighing matrices with added two-level categorical factors. *Journal of Quality Technology*, **52**(2): 168 – 181. <https://doi.org/10.1080/00224065.2019.1571341>
- Bailey, R. A. (1985). Factorial design and Abelian groups. *Linear Algebra and its Applications*, **70**: 349 – 368.
- Bate, S. T. and Chatfield, M. J. (2016a). Identifying the structure of the experimental design. *Journal of Quality Technology*, **48**(4): 343 – 364. <https://doi.org/10.1080/00224065.2016.11918173>
- Bate, S. T. and Chatfield, M. J. (2016b). Using the structure of the experimental design and the randomization to construct a mixed model. *Journal of Quality Technology*, **48**(4): 365 – 387. <https://doi.org/10.1080/00224065.2016.11918174>
- Chen, J. (1998). Intelligent search for 2^{13-6} and 2^{14-7} minimum aberration designs. *Statistica Sinica*, **8**: 1265 – 1270.
- Chen, H. and Hedayat, A.S. (1998). Some recent advances in minimum aberration designs. *Lecture Notes – Monograph Series: New Developments and Applications in Experimental Design*, **34**, 186 – 198.
- Cheng, C.-S. and Mukerjee, R. (1998). Regular fractional factorial designs with minimum aberration and maximum estimation capacity. *Annals of Statistics*, **26**: 2289 – 2300.
- Ekhosuehi, V. U., Iruwegbukpe, D. O. and Mbegbu, J. I. (2018). The minimum aberration criterion for screening experiments at two levels from an entropy-based perspective. *Communications in Statistics – Theory and Methods*, **47**(8): 1868 – 1881.
- Fries, A. and Hunter, W.G. (1980). Minimum aberration 2^{k-p} designs. *Technometrics*, **22**: 601 – 608.
- Georgiou, S. D., Stylianou, S. and Aggarwal, M. (2014). Efficient three-level screening designs using

- weighing matrices. *Statistics: A Journal of Theoretical and Applied Statistics*, **48**(4): 815 – 833. <https://doi.org/10.1080/02331888.2012.760097>
- Herstein, I.N. (1975). *Topics in Algebra* (2nd ed.). New York: John Wiley & Son.
- Hu, J. and Zhang, R. (2009). Maximal rank minimum aberration and doubling. *Statistics and Probability Letters*, **79**: 915 – 919.
- Liao, C.-T. and Chai, F.-S. (2009). Design and analysis of two-level factorial experiments with partial replication. *Technometrics*, **51**: 66 – 74.
- Mee, R.W. (2009). *A comprehensive guide to factorial two-level experimentation*. New York: Springer Science.
- Montgomery, D. C. and Jennings, C. L. (2006). An overview of industrial screening experiments. In: A. Dean and S. Lewis (eds), *Screening*. Springer, New York. https://doi.org/10.1007/0-387-28014-6_1.
- Olsen, R., Lawson, J., Rohbock, N. and Woodfield, B. (2016). Practical comparison of traditional and definitive screening designs in chemical process development. *International Journal of Experimental Design and Process Optimisation*, **5**(1/2): 1 – 22.
- Pistone, G., Riccomagno, E. and Rogantin, M. P. (2009). Methods in algebraic statistics for the design of experiments. In: L. Pronzato and A. Zhigljavsky (eds), *Optimal Design and Related Areas in Optimization and Statistics*. Springer Optimization and Its Applications, **28**. Springer, New York. https://doi.org/10.1007/978-0-387-79936-0_5.
- Santamaria, J. L., Valentin, V. and Huerta, G. (2019). Quantifying the effect of construction site factors on concrete compressive strength using designed experiments. *International Journal of Experimental Design and Process Optimisation*, **6**(1): 26 – 49. doi: 10.1504/IJEDPO.2019.097466
- Tang, B. and Wu, C.F.J. (1996). Characterization of minimum aberration 2^{n-k} designs in terms of their complementary designs. *Annals of Statistics*, **24**: 2549 – 2559.
- Vicente, G., Coteron, A., Martinez, M. and Aracil, J. (1998). Application of the factorial design of experiments and response surface methodology to optimise biodiesel production. *Industrial Crops and Products*, **8**: 29 – 35.
- Wu, C.F.J. and Hamada, M. (2000). *Experiments: Planning, analysis and parameter design optimization*. New York: Wiley.