BENIN JOURNAL OF
STATISTICS
ISSN 2682-5767
Vol. 3, pp. 41– 54 (2020)

Beta-Odd Generalized Exponential Family of Distribution

S. M. Umar[1*] and F. I. Zakari[2]
[1,2] *Department of Mathematical Science, Bayero University Kano, PMB 3011, Kano, Nigeria*

Abstract. In this research work, we introduce a new family of distributions termed the Beta Odd Generalized Exponential (BOGE) distribution. Various properties of the model are derived. We present and study three special cases of the BOGE family of distribution. Estimation for the parameters of the new distribution are discussed by the method of maximum likelihood. Applications to two real data sets are provided in order to demonstrate the performance of the proposed family of distributions which shows that it is better than some existing distributions.

Keywords: Odd generalized exponential distribution, moments, maximum likelihood estimates.

Published by: Department of Statistics, University of Benin, Nigeria

1. Introduction

Parametric statistical inferences and modeling of data sets require the knowledge of appropriate distributional assumptions of the data sets. Thus, classical statistical distributions have been used in many areas of applied and social sciences to make inferences and model real life data. Hence, modeling the real life data with some existing classical distributions does not provide a reasonable parametric fit and is often an approximation rather than reality. An alternative approach to overcome these challenges is to use nonparametric methods to model the data sets since they do not depend on distributional assumptions like the parametric methods. However, the non-parametric methods have their own drawbacks. This includes loss in power when the parametric method is appropriate, lack of imprecision measurement, computational difficulties, difficult to calculate residual variability and loss information.

The art of proposing generalized class of distributions by extending the common families of continuous distribution has attracted theoretical and applied statisticians due to their flexible properties. These new families of distributions have been used for modelling data in many applied areas such as modeling machine life cycle in engineering, modeling duration without claims in actuarial science, modeling survival times of patients after surgery in the medical science, modeling failure rate of software in computer science, average time from marriage to divorce in the social science and modeling environmental pollution in environmental sciences.

The techniques for modifying the classical distributions are usually referred to as generators in literature and are capable of improving the goodness-of-fit of the modified distributions. Eugene et al. (2002) introduced a general class of distributions generated from the logit of the beta random variable of which beta-normal distribution is a special case of the family and called it the Beta-G family of distributions. According to them, the cumulative distribution function of the Beta-G family is defined as

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{a-1}(1-w)^{b-1} dw, \tag{1}$$

where $a, b > 0$, are two additional parameters, $G(x)$ is the cdf of a random variable X and $B(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1} dw$ is the beta function.

[*] Corresponding author. Email: surajoumar@yahoo.com

Tahir et al. (2015) defined the cumulative distribution function of a new family of distribution by replacing x in the generalized exponential model by the odd function $G(x;\xi)/\bar{G}(x;\xi)$ termed as the Odd Generalized Exponential(ODE) family of distributions as

$$F(x) = F(x;\alpha,\lambda,\xi) = \left(1 - e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x,\xi)}}\right)^{\alpha}, \quad x > 0, \tag{2}$$

where $G(x;\xi)$ is the cumulative distribution function of the baseline distribution.

Gupta et al. (1998) introduced a new family of distributions namely Exponentiated Exponential distribution and Gupta et al. (2001) studied some properties of the new family. Eugene et al. (2002) introduced a general class of distributions generated from the logit of the beta random variable of which beta-normal distribution is a special case of the family. Nadarajah and Kotz (2004) introduced the beta Gumbel distribution, Nadarajah and Kotz (2005) the beta exponential distribution, Barreto-Souza et al. (2010) the beta generalized exponential (BGE) distribution and Singla el al. (2012) the Beta Generalized Weibull (BGW) distribution.

Tahir et al. (2015) proposed a new family of continuous distributions called the odd generalized exponential (OGE) family, whose hazard rate could be increasing, decreasing, J, reversed-J, bathtub and upside-down bathtub.

Muhammad (2016) introduced a new family of distributions called the Poisson-odd generalized exponential distribution (POGE) which has the odd generalized exponential as its limiting distribution and derived Various properties of the new model. A new class of distributions called the generalized odd generalized exponential family was introduced by Alizadeh et al. (2017b) and some of its mathematical properties including explicit expressions for moments, quantile and generating functions, Renyi, Shannon and q-entropies, order statistics and probability weighted moments were derived.

In this paper we propose a new family of distribution by choosing the baseline distribution of the Beta G distribution to be the Odd Generalized exponential family of distribution.

2. Materials and method

Consider the cumulative distribution function (cdf) of a random variable X given by Eugene et al. (2002) which they defined a class of generalized distributions, the "beta G" distribution as

$$F(x) = \frac{1}{B(a,b)}\int_0^{M(x)} w^{a-1}(1-w)^{b-1}dw, \tag{3}$$

where $a, b > 0$, are two additional parameters and $B(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}dw$ is the beta function. We can rewrite (3) as

$$F(x) = I_{M(x)}(a,b), \tag{4}$$

where $I_y(a,b) = B(a,b)^{-1}\int_0^y w^{a-1}(1-w)^{b-1}dw$ denotes the incomplete beta function ratio, that is the cdf of beta distribution with parameters a and b.

The probability density function (pdf) corresponding to (3) can be written in the form

$$f(x) = \frac{1}{B(a,b)}[M(x)]^{a-1}(1-M(x))^{b-1}m(x). \tag{5}$$

We now consider the baseline cumulative distribution function (cdf) $M(x;\xi)$ and probability density function (pdf) $m(x;\xi)$ depending on a parameter vector ξ, where $\xi = (\xi_1, \xi_2, \ldots)$ to be the Odd generalized exponential family of distribution given by Tahir et al. (2015) respectively as

$$M(x) = M(x;\alpha,\lambda,\xi) = \left(1 - e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x,\xi)}}\right)^{\alpha}, \quad x > 0 \tag{6}$$

and

$$m(x)=m(x;\alpha,\lambda,\xi)=\frac{\alpha\lambda g(x;\xi)}{\bar{G}(x,\xi)^2}e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x,\xi)}}\left(1-e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x,\xi)}}\right)^{\alpha-1},\quad x>0. \tag{7}$$

3. Density and distribution functions

We now introduce the distribution called Beta odd generalized exponential (BOGE) family of distribution by choosing the baseline distribution $M(x)$ cdf (6) and pdf (7) of OGE family of distribution into the cdf (3) and pdf (5) of the Beta G family respectively. The cdf is given as

$$F(x)=\frac{1}{B(a,b)}\int_0^{\left(1-e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x,\xi)}}\right)^{\alpha}} w^{a-1}(1-w)^{b-1}dw,\quad x>0, \tag{8}$$

where $\alpha,\lambda,a,b>0$ are parameters, $G(x;\xi)$ is the baseline cdf, ξ a vector parameter, $\bar{G}(x;\xi)=1-G(x;\xi)$ and $B(a,b)=\int_0^1 w^{a-1}(1-w)^{b-1}dw$ is the beta function. The cdf (8) can be rewritten as

$$F(x)=I_{\left(1-e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\alpha}}(a,b), \tag{9}$$

where $I_y(a,b)=B(a,b)^{-1}\int_0^y w^{a-1}(1-w)^{b-1}dw$ denotes the incomplete beta function ratio, i.e., the cdf of the beta distribution with parameters a and b. The pdf corresponding to (8) is given as

$$\begin{aligned}f(x)=&\frac{1}{B(a,b)}\left(1-e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\alpha(a-1)}\left\{1-\left(1-e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\alpha}\right\}^{b-1}\\&\times\frac{\alpha\lambda g(x;\xi)}{\bar{G}(x;\xi)^2}e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\left(1-e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x,\xi)}}\right)^{\alpha-1},\end{aligned}$$

which reduces to

$$f(x)=\frac{\alpha\lambda}{B(a,b)}\frac{g(x;\xi)}{\bar{G}(x;\xi)^2}e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\left(1-e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\alpha a-1}\left\{1-\left(1-e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\alpha}\right\}^{b-1},\quad x>0, \tag{10}$$

which can be expressed in mixture form in terms of cdfs of the GE distributions as

$$f(x)=\frac{\alpha\lambda}{B(a,b)}\frac{g(x;\xi)}{\bar{G}(x;\xi)^2}e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}G_{\lambda,\alpha a-1}\left(\frac{G(x;\xi)}{\bar{G}(x,\xi)}\right)\left\{1-G_{\lambda,\alpha}\left(\frac{G(x;\xi)}{\bar{G}(x,\xi)}\right)\right\}^{b-1}, \tag{11}$$

where $G_{\lambda,\alpha}\left(\frac{G(x)}{\bar{G}(x)}\right)=\left(1-e^{-\lambda\frac{G(x)}{\bar{G}(x)}}\right)^{\alpha}$ is the cdf of Generalized exponential distribution with parameter λ and power parameter $\alpha>0$.

The following are some existing members of BOGE family of distribution:

(1) When $\alpha=a=b=1$ and $G=\frac{x}{1+x}$, the BOGE reduces to exponential distribution.
(2) When $a=b=1$, the BOGE reduces to the OGE family of distribution by Tahir et al (2015).
(3) When $G=\frac{x}{1+x}$, the BOGE reduces to the BGE distribution by Barreto-Souza et al (2010).
(4) When $G=\frac{x}{1+x}$ and $\alpha=1$, the BOGE reduces to the BE distribution by Nadarajah and Kotz (2005).

The hazard rate function of the BOGE family of distribution is given by

$$h(x)=\frac{\alpha\lambda g(x;\xi)e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\left(1-e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\alpha a-1}\left\{1-(1-e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}})^{\alpha}\right\}^{b-1}}{B(a,b)\bar{G}(x;\xi)^2 I_{1-\left(1-e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x,\xi)}}\right)^{\alpha}}(a,b)} \tag{12}$$

To test the validity of the pdf, we use the fact that

$$\int_0^\infty f(x)dx = 1 \tag{13}$$

substituting the pdf (10) into equation (13) gives

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{\alpha\lambda}{B(a,b)} \frac{g(x;\xi)}{[\bar{G}(x;\xi)]^2} e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}} \left(1 - e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\alpha a-1} \left\{1 - \left(1 - e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\alpha}\right\}^{b-1} dx. \tag{14}$$

Let $u = 1 - e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}$, then $dx = \frac{[\bar{G}(x;\xi)]^2}{\lambda g(x)e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}}du$. We have

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{\alpha\lambda}{B(a,b)} \frac{g(x;\xi)}{[\bar{G}(x;\xi)]^2} e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}} u^{\alpha a-1} \left(1 - u^{\alpha}\right)^{b-1} \frac{[\bar{G}(x;\xi)]^2}{\lambda g(x;\xi)e^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}}du. \tag{15}$$

which gives

$$\int_0^\infty f(x)dx = \frac{\alpha}{B(a,b)} \int_0^1 u^{\alpha a-1}(1-u^\alpha)^{b-1}du \tag{16}$$

We then let $v = u^\alpha$, then $du = \frac{dv}{\alpha u^{\alpha-1}}$

$$\begin{aligned}
\int_0^\infty f(x)dx =& \frac{\alpha}{B(a,b)} \int_0^1 u^{\alpha a-1}(1-v)^{b-1}\frac{dv}{\alpha u^{\alpha-1}} \\
=& \frac{1}{B(a,b)} \int_0^1 u^{\alpha a-\alpha}(1-v)^{b-1}dv \\
=& \frac{1}{B(a,b)} \int_0^1 u^{\alpha(a-1)}(1-v)^{b-1}dv \\
=& \frac{1}{B(a,b)} \int_0^1 v^{a-1}(1-v)^{b-1}dv \\
=& \frac{1}{B(a,b)} B(a,b) = 1.
\end{aligned} \tag{17}$$

This gives the validation of BOGE as pdf.

3.1 *Special cases of BOGE distribution.*

In this subsection, we present some special cases of the BOGE family of distribution namely the Beta odd generalized exponential - Half logistic (BOGE - HL), Beta odd generalized exponential - Exponential (BOGE-E) and Beta odd generalized exponential - Uniform (BOGE-U) distributions which are very useful in solving various problems in practical applications in the fields of sciences and applied sciences.

3.1.1 The BOGE - half logistic (BOGE-HL) distribution.

The Beta odd generalized exponential - Half logistic (BOGE-HL) distribution is obtained by choosing the baseline cdf and pdf in (8) and (10) to be the Half logistic distribution defined by $G(x) = \frac{(1-e^{-x})}{(1+e^{-x})}$ and $g(x) = \frac{2e^{-x}}{(1+e^{-x})^2}$ respectively. For $x > 0$ and parameters $\alpha, \lambda, a, b > 0$, the cdf and pdf of the BOGE-HL distribution are given respectively by

$$F(x;\alpha,\lambda,a,b) = \frac{1}{B(a,b)} \int_0^{\left(1-e^{-\lambda\left(\frac{e^x-1}{2}\right)}\right)^\alpha} w^{a-1}(1-w)^{b-1}dw \tag{18}$$

$$f(x;\alpha,\lambda,a,b)=\frac{\alpha\lambda e^{x-\lambda\left(\frac{e^{x}-1}{2}\right)}}{2B(a,b)}\left(1-e^{-\lambda\left(\frac{e^{x}-1}{2}\right)}\right)^{\alpha a-1}\left\{1-\left(1-e^{-\lambda\frac{e^{x}-1}{2}}\right)^{\alpha}\right\}^{b-1} \tag{19}$$

Figures 1 – 3 display, respectively, the plots of the pdf, cdf and hrf of the BOGE-HL distribution for some selected parameter values.

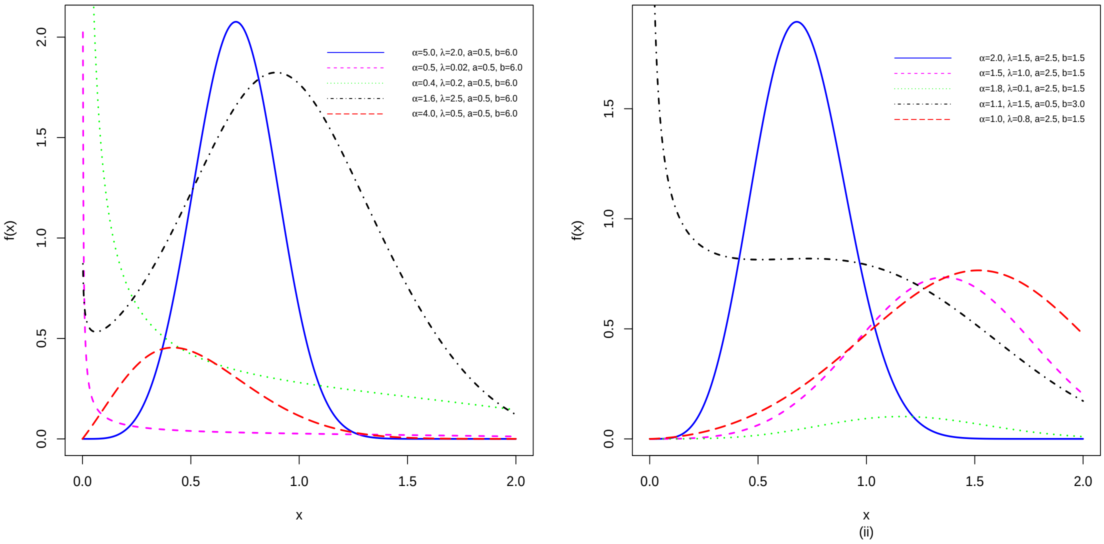

Figure 1: Fitted pdf of the BOGE-HL for some selected values

3.1.2 The BOGE - Exponential (BOGE-E) distribution.

The Beta Odd Generalized Exponential - Exponential (BOGE-E) distribution is obtained by choosing the baseline cdf and pdf in (8) and (10) to be the exponential distribution defined by $G(x;\theta)=1-e^{-\theta x}$ and

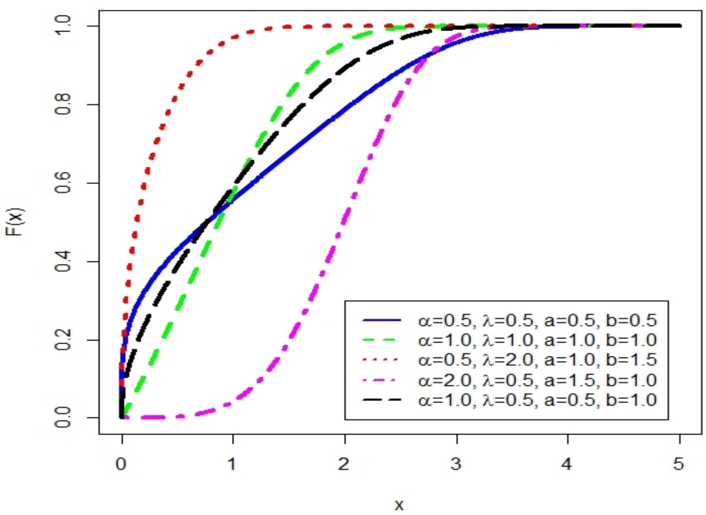

Figure 2: Fitted cdf of the BOGE-HL for some selected values

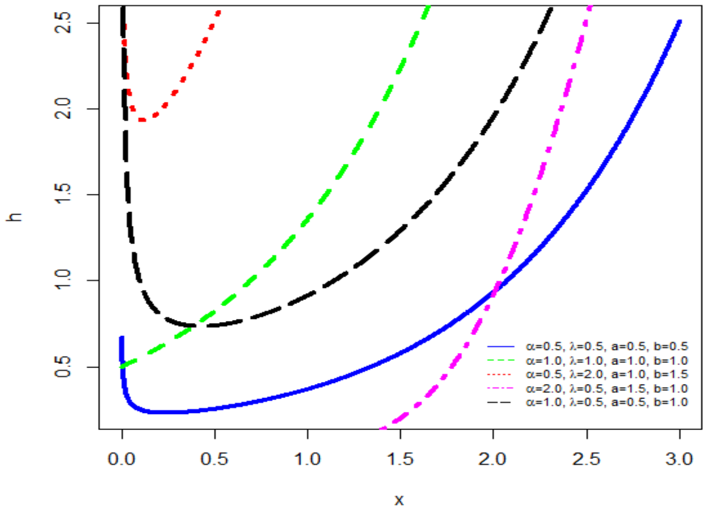

Figure 3: Fitted hrf of the BOGE-HL for some selected values

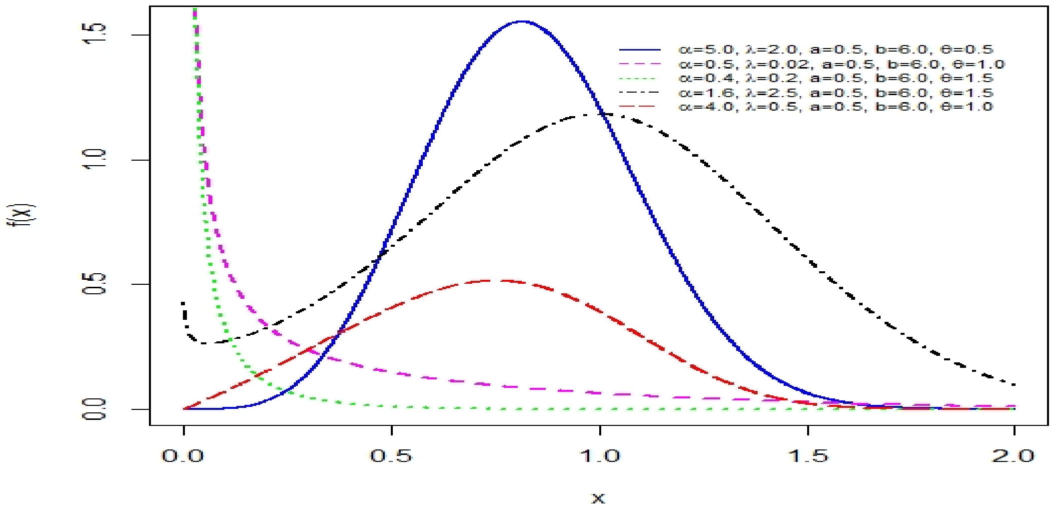

Figure 4: Fitted pdf of the BOGE-E for some selected values

$g(x;\theta) = \theta e^{-\theta x}$ respectively. For $x > 0$ and parameters $\alpha, \lambda, \theta, a, b > 0$, the cdf and pdf of the BOGE-E distribution are given respectively by

$$F(x;\alpha,\lambda,\theta,a,b) = \frac{1}{B(a,b)} \int_0^{\left(1-e^{-\lambda\left(e^{\theta x}-1\right)}\right)^{\alpha}} w^{a-1}(1-w)^{b-1}dw, \tag{20}$$

$$f(x;\alpha,\lambda,\theta,a,b) = \frac{\alpha\lambda\theta e^{\theta x}}{B(a,b)} e^{-\lambda(e^{\theta x}-1)} \left(1-e^{-\lambda(e^{\theta x}-1)}\right)^{\alpha a-1} \left\{1-\left(1-e^{-\lambda(e^{\theta x}-1)}\right)^{\alpha}\right\}^{b-1} \tag{21}$$

Figure 4 displays the plot of the pdf of the BOGE-E distribution for some selected parameter values.

3.2 *Expression of the BOGE distribution in the form of series*

We provide a series representation of the BOGE distribution based on certain conditions.

LEMMA 3.1 *The pdf of the BOGE family of distributions can be express as the density function of the exponentiated distribution as*

$$f(x) = \sum_{i,j=0}^{\infty} \sum_{k,l=0}^{\infty} \varrho_{(i,j,k,l)} g^*(x; k+l+1, \zeta) \tag{22}$$

where

$$\varrho_{(i,j,k,l)} = \binom{b-1}{i} \binom{\alpha(a+i)-1}{j} \binom{-(k+2)}{l} \frac{\alpha(-1)^{i+j+k+l} \lambda^{k+1} (j+1)^k}{B(a,b) k! (k+l+1)}$$

and $g^*(x; k+l+1, \zeta)$ *is the density of exponentiated* $G(x; \zeta)$ *to the power of* $k+l+1$.

Proof. First for $|z| < 1$ and for $a > 0$ real and non-integer, we have the series expansion of $(1-z)^{a-1}$ as

$$(1-z)^{a-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(a)}{\Gamma(a-j) j!} z^j = \sum_{j=0}^{\infty} \binom{a-1}{j} (-1)^j z^j. \tag{23}$$

Applying the series expansion in (23) and the exponential expansion into equation (10), we arrive at

$$f(x) = \sum_{i,j=0}^{\infty} \sum_{k,l=0}^{\infty} \binom{b-1}{i} \binom{\alpha(a+i)-1}{j} \binom{-(k+2)}{l} \frac{\alpha(-1)^{i+j+k+l} \lambda^{k+1} (j+1)^k}{B(a,b) k! (k+l+1)} g^*(x; k+l+1, \zeta) \tag{24}$$

and finally we arrive at

$$f(x) = \sum_{i,j=0}^{\infty} \sum_{k,l=0}^{\infty} \varrho_{(i,j,k,l)} g^*(x; k+l+1, \zeta) \tag{25}$$

■

4. Some properties of the BOGE distribution

Some important properties of the BOGE distribution are going to be presented in this section

4.1 *Quantile*

The quantile function is a useful measure for describing the distribution of a random variable. It plays a key role when simulating random numbers and can also be used to compute the median, kurtosis and skewness of the distribution of a random variable. The quantile function of the BOGE family of distribution is obtained by inverting $F(x) = u$, where $F(x)$ is the cdf of the BOGE family of distribution given by equation (9), and u to be a uniform variate on the interval $[0, 1]$. That is we solve a solution to the equation given by

$$I_{\left(1-e^{-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\alpha}}(a,b) = u, \tag{26}$$

taking the inverse of the incomplete beta function ratio and then the logarithm, we get

$$G(x;\xi) = \frac{-\frac{1}{\lambda} \log\left(1 - [I_u^{-1}(a,b)]^{\frac{1}{\alpha}}\right)}{1 - \frac{1}{\lambda} \log\left(1 - [I_u^{-1}(a,b)]^{\frac{1}{\alpha}}\right)}, \tag{27}$$

consequently

$$Q(u) = G^{-1}\left[\frac{-\frac{1}{\lambda}\log\left(1-[I_u^{-1}(a,b)]^{\frac{1}{\alpha}}\right)}{1-\frac{1}{\lambda}\log\left(1-[I_u^{-1}(a,b)]^{\frac{1}{\alpha}}\right)}\right] \tag{28}$$

which is the quantile function of the BOGE family of distributions.

4.2 *Moment*

It is imperative to derive the moments when a new distribution is proposed. They play a significant role in statistical analysis, particularly in applications. Moments are used in computing measures of central tendency, dispersion and shapes among others.

PROPOSITION 4.1 *The rth non-central moment of the BOGE family of distribution is given by*

$$E(X^r) = \sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty} \varrho_{(i,j,k,l)} E(Y_e^r) \tag{29}$$

where $E(Y_e^r)$ is the r^{th} moment of the exponentiated $G(x;\zeta)$ distribution with power parameter $k+l+1$.

Proof. For a random variable $X \sim BOGE(\xi)$, the r^{th} moment of X is obtained from the relation

$$E(X^r) = \int_0^{\infty} x^r f(x)dx. \tag{30}$$

Substituting equation (22) into equation (30), we have

$$E(X^r) = \sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty} \varrho_{(i,j,k,l)} E(Y_e^r), \tag{31}$$

which is the non-central moment of the BOGE family of distribution. ■

4.3 *Moment generating function*

The moment generating function (mgf) of the BOGE distribution can be computed from

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r), \tag{32}$$

substituting (31) in (32) gives

$$M_X(t) = \sum_{i,j,k=0}^{\infty}\sum_{l,r=0}^{\infty} \frac{t^r}{r!} \varrho_{(i,j,k,l)} E(Y_e^r), \tag{33}$$

which is the mgf of the BOGE distribution.

5. Order statistics

In statistical theory and applications, the order statistic is one of the most essential fundamental tools. In order to derive the distribution of the i^{th} order statistics from the BOGE family distribution, let $X_1, X_2, \cdots, X_n$ be independently and identically distributed (iid) random variables from the BOGE pdf, $f(x)$ and $X_1 < X_2 < \cdots < X_n$ to denote the corresponding order statistics. The pdf $f_{i:n}(x)$ of the i^{th} order statistic is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i}, \tag{34}$$

$$f_{i:n}(x) = \sum_{p=0}^{n-i} \frac{(-1)^p n!}{(i-1)!(n-i-p)!p!} f(x)[F(x)]^{p+i-1}. \tag{35}$$

Substituting the closed forms of the cdf $F(x)$ and the pdf $f(x)$ equations (9) and (22), respectively, into (35) and some algebraic manipulations we arrive at

$$f_{i:n}(x) = \sum_{p=0}^{n-i} \sum_{i,j=0}^{\infty} \sum_{k,l=0}^{\infty} \varrho_{(i,j,k,l,p)} g^*(x; k+l+1, \zeta) \left[I_{\left(1-e^{-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\alpha}}(a,b) \right]^{p+i-1} \tag{36}$$

which is the distribution of the order statistics, where

$$\varrho_{(i,j,k,l,p)} = \binom{b-1}{i} \binom{\alpha(a+i)-1}{j} \binom{-(k+2)}{l} \frac{\alpha \lambda^{k+1} (-1)^{i+j+k+l+p} (j+1)^k n!}{B(a,b) k! (i-1)!(n-i-p)!p!}$$

6. Estimation

In this section, we estimate the unknown parameters of the BOGE family of distribution by the method of maximum likelihood estimation. Let $X_1, X_2, \cdots, X_n$ be a random sample each from the BOGE family of distributions as defined in (10) of size n independently and identically distributed random variables with parameters α, λ, a, b and ξ with observed values $x_1, x_2, \cdots, x_n$. Let $\theta = (\alpha, \lambda, a, b, \xi^T)^T$ be a $(p+4) \times 1$ parameter vector, where ξ is a $(p \times 1)$ baseline parameter vector. For determining the MLE of θ, the likelihood function $L(\theta)$ is given by

$$\begin{aligned} L(\theta) = \prod_{i=1}^{n} f(x_i) &= \prod_{i=1}^{n} \left[\frac{\alpha \lambda}{B(a,b)} \frac{g(x_i;\xi)}{\bar{G}(x_i;\xi)^2} e^{-\lambda H(x_i;\xi)} \left(1 - e^{-\lambda H(x_i;\xi)}\right)^{\alpha a - 1} \left\{1 - \left(1 - e^{-\lambda H(x_i;\xi)}\right)^{\alpha}\right\}^{b-1} \right] \\ &= \frac{\alpha^n \lambda^n}{(B(a,b))^n} \prod_{i=1}^{n} \left[\frac{g(x_i;\xi)}{\bar{G}(x_i;\xi)^2} e^{-\lambda H(x_i;\xi)} \left(1 - e^{-\lambda H(x_i;\xi)}\right)^{\alpha a - 1} \left\{1 - \left(1 - e^{-\lambda H(x_i;\xi)}\right)^{\alpha}\right\}^{b-1} \right]. \end{aligned}$$

Taking the $\log$ of the likelihood function $L(\theta)$, we have

$$\begin{aligned} \log L(\theta) =& n \log \alpha + n \log \lambda - n \log B(a,b) \ + \prod_{i=1}^{n} \left[\log g(x_i, \xi) - 2 \log \bar{G}(x_i, \xi) \right] \\ &+ \prod_{i=1}^{n} \left[-\lambda H(x_i, \xi) + (\alpha a - 1) \log \left(1 - e^{-\lambda H(x_i;\xi)}\right) + (b-1) \log \left\{1 - \left(1 - e^{-\lambda H(x_i;\xi)}\right)^{\alpha}\right\} \right] \end{aligned}$$

Let

$$\begin{aligned}\ell = \log L(\theta) =& n\log\alpha + n\log\lambda - n\log B(a,b) + \sum_{i=1}^{n}\log g(x_i,\xi) - 2\sum_{i=1}^{n}\log\bar{G}(x_i,\xi) - \lambda\sum_{i=1}^{n}H(x_i,\xi)\\ &+ (\alpha a - 1)\sum_{i=1}^{n}\log\left(1 - e^{-\lambda H(x_i;\xi)}\right) + (b-1)\sum_{i=1}^{n}\log\left\{1 - \left(1 - e^{-\lambda H(x_i;\xi)}\right)^{\alpha}\right\}\end{aligned} \tag{37}$$

where $H(x_i;\xi) = \frac{G(x_i;\xi)}{\bar{G}(x_i;\xi)}$. The components of the score vector, $U(\theta) = \frac{\partial\ell}{\partial\theta} = \left(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\xi}\right)^T$, are given by

$$\begin{aligned}U_\alpha = \frac{\partial\ell}{\partial\alpha} =& \frac{n}{\alpha} + a\sum_{i=1}^{n}\log(1-z_i) + (b-1)\sum_{i=1}^{n}\frac{(1-z_i)^{\alpha}\log(1-z_i)}{1-(1-z_i)^{\alpha}}\\ U_\lambda = \frac{\partial\ell}{\partial\lambda} =& \frac{n}{\lambda} - \sum_{i=1}^{n}H(x_i;\xi) + (\alpha a-1)\sum_{i=1}^{n}\frac{H(x_i;\xi)z_i}{(1-z_i)} + \alpha(b-1)\sum_{i=1}^{n}\frac{(1-z_i)^{\alpha-1}H(x_i;\xi)z_i}{1-(1-z_i)^{\alpha}}\\ U_a = \frac{\partial\ell}{\partial a} =& n\left[\Psi(a+b) - \Psi(a)\right] + \alpha\sum_{i=1}^{n}\log(1-z_i)\\ U_b = \frac{\partial\ell}{\partial b} =& n\left[\Psi(a+b) - \Psi(b)\right] + \sum_{i=1}^{n}\log\left\{1-(1-z_i)^{\alpha}\right\}\\ U_\xi = \frac{\partial\ell}{\partial\xi} =& \sum_{i=1}^{n}\frac{g'(x_i;\xi)}{g(x_i;\xi)} - 2\sum_{i=1}^{n}\frac{G'(x_i;\xi)}{G(x_i;\xi)} - \lambda\sum_{i=1}^{n}H'(x_i;\xi) + (\alpha a-1)\sum_{i=1}^{n}\frac{\lambda H'(x_i;\xi)}{1-z_i}\\ &- \alpha\lambda(b-1)\sum_{i=1}^{n}\frac{(1-z_i)^{\alpha-1}H'(x_i;\xi)z_i}{1-(1-z_i)^{\alpha}}\end{aligned}$$

where $z_i = e^{-\lambda H(x_i;\xi)}$, $g'(x_i;\xi) = \frac{\partial g(x_i;\xi)}{\partial\xi}$ and $G'(x_i;\xi) = \frac{\partial G(x_i;\xi)}{\partial\xi}$. Setting the non linear system of equations $U_\lambda = U_\alpha = U_a = U_b = U_\xi = 0$ and solving them simultaneously yields the the MLE $\widehat{\theta} = (\widehat{\alpha}, \widehat{\lambda}, \widehat{a}, \widehat{b}, \widehat{\xi}^T)^T$. To solve these equations, it is usually more convenient to use non linear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ.

7. Application

In this section, we present the application of the BOGE distribution using two real data sets. For comparison, we fitted the first data set with the BOGE-HL distribution and for the second data we fitted the BOGE-E distribution.

7.1 *Glass Fibres Data*

The data set is obtained from Smith and Naylor (1987) and recently Maiti and Pramanik (2015) fitted the data for the Odds Generalized Exponential - Exponential Distribution. The data consists of 63 observations of the strengths of 1.5cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the units of the measurement are not given in the paper. The data set is given in Table 1.

Table 2 shows the maximum likelihood estimates (MLEs) associated to each distribution fitted to the dataset. It also shows the log-likelihood ℓ, the AIC and the BIC for each model. The AIC and the BIC of the BOGE-HL can be seen to be lower when compared to Beta generalized exponential (BGE), Odd generalized weibull (OE-W), Odd exponential normal (OE-N), Odd generalized exponential half logistic (OGE-HL), Beta exponential (BE), Generalized exponential (GE) and exponential (E) distributions, indicating that the BOGE-HL outperforms all the other models for fitting the dataset.

Table 1: Glass Fibres Data

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73	1.81	2.00
0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01
0.77	1.11	1.28	1.42	1.50	1.54	1.60	1.62	1.66	1.69	1.76	1.84	2.24
0.81	1.13	1.29	1.48	1.50	1.55	1.61	1.62	1.66	1.70	1.77	1.84	0.84
1.24	1.30	1.48	1.51	1.55	1.61	1.63	1.67	1.70	1.78	1.89		

Table 2: MLEs, ℓ, AIC and BIC for the first data

Model	α	λ	a	b	β	ℓ	AIC	BIC
BOGE-HL	6.3051	0.3710	0.5496	23.5593	-	-14.1725	36.5448	45.1173
OGEED	3.6474	0.0024	-	-	-	-14.8100	36.6160	40.9760
BGE	22.6124	0.92271	0.4125	93.4655	-	-15.5995	39.1990	47.7715
OE - W	-	0.0721	-	-	1.9603	-16.4613	36.9227	41.2088
OE - N	-	0.0121	-	-	0.7385	-17.5979	39.1958	43.4820
OGE - HL	1.3840	7.3084	-	-	-	-19.9672	43.93442	48.22069
BE	-	0.3898	17.7706	22.7222	-	-24.1270	54.2540	60.6834
GE	31.3032	2.6105	-	-	-	-31.3834	66.7668	71.0531
E	-	0.644	-	-	-	-88.8300	179.660	181.8031

Figure 5 illustrates the histogram for the data set and the fitted BOGE-HL distribution. The R code used in computing the estimates and log-likelihood function is given in Appendix A.

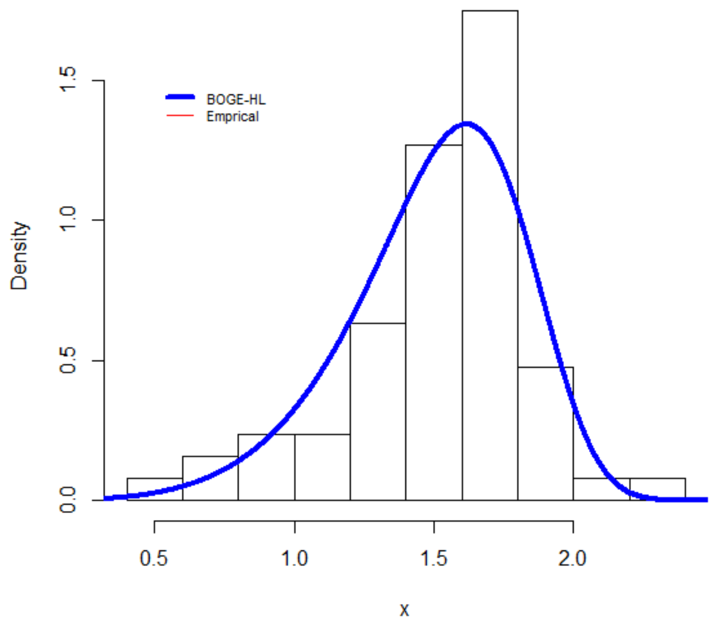

Figure 5: Fitted pdf of the BOGE-HL for the first data

7.2 *Precipitation Data*

The data was first reported by Hinkley (1977) and consists of 30 observations of March precipitation (in inches) in Minneapolis/ St Paul. Nasiru (2018) fitted the data for the New Exponentiated Generalized Modified Inverse Rayleigh (NEGMIR). The data set is given in Table 3.

Table 4 shows the maximum likelihood estimates (MLEs) associated to each distribution fitted to the dataset. It also shows the log-likelihood ℓ, the AIC and the BIC for each model. The AIC and the BIC of

Table 3: Precipitation Data

0.77	1.74	0.81	1.20	1.95	1.20	0.47	1.43	3.37	2.20
3.00	3.09	1.51	2.10	0.52	1.62	1.31	0.32	0.59	0.81
2.81	1.87	1.18	1.35	4.75	2.48	0.96	1.89	0.90	2.05

the BOGE-E can be seen to be lower when compared to New Exponentiated Generalized Modified Inverse Rayleigh (NEGMIR) Distribution, Exponentiated Generalized Modified Inverse Rayleigh (EGMIR), New exponential Generalized Inverse Rayleigh (NEGIR) and New exponential Generalized Inverse Exponential (NEGIE), indicating that the BOGE-E outperforms all the other models for fitting the dataset.

Table 4: MLEs, ℓ, AIC and BIC for the second data

Model	α	λ	θ	a	b	ℓ	AIC	BIC
BOGE-E	0.322	4.929	0.018	14.082	13.380	-37.040	84.079	91.085
NEGMIR	3.022	0.225	2.246	0.112	24.039	-37.870	85.738	92.744
EGMIR	1.658	-	2.918	0.235	1.877	-42.750	93.492	99.097
NEGIR	-	0.087	1.305	0.219	10.813	-40.210	88.421	94.025
NEGIE	9.708	8.228	-	0.258	0.092	-40.460	88.912	94.517

Figure 6 illustrates the histogram for the data set and the fitted BOGE-E distribution.

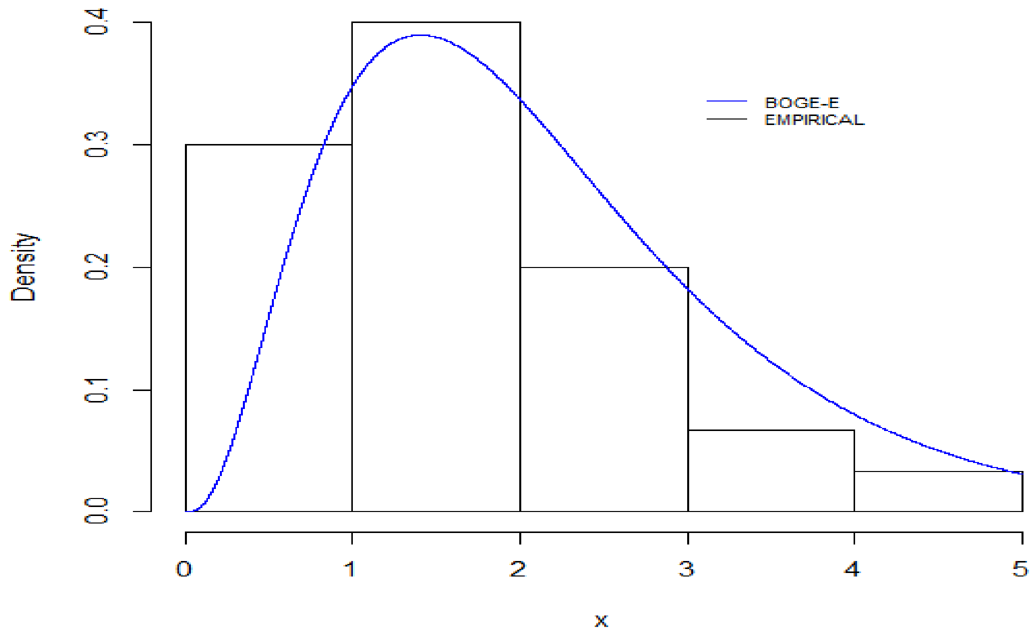

Figure 6: Fitted pdf of the BOGE-E for the second data

8. Conclusion

In this research work, we introduced a four parameter family of distribution known as the Beta-Odd Generalized Exponential (BOGE) family of distribution. It generates by choosing the baseline distribution of the Beta-G distribution to be the Odd Generalized Exponential (OGE) family of distribution. Properties of the proposed distribution such as the quantile function, hazard rate function, moment, moment generating function and order statistics were derived and the maximum likelihood estimation method was used for estimating the parameters.

References

Alizadeh, M., Cordeiro, G. M., Pinho, L. G. B. and Ghosh, I. (2017). The Gompertz-G family of distributions. Journal of Statistical Theory and Practice. http://dx.doi.org/10.1080/15598608.2016.1267668.

Alizadeh, M., Ghosh, I., Yousof, H. M., Rasekhi, M. and Hamedani, G. G. (2017). The generalized odd generalized exponential family of distributions: properties, characterizations and application. Journal of Data Science, **16**: 443-466.

Akaike, A. (1973). Information theory and an extension of the maximum likelihood principle. In International Symposium on Information Theory, 2nd, Tsahkadsor, American SSR. 267-281.

Barreto-Souza, W., Santos, A. H. S. and Cordeiro, G. M. (2010). The beta generalized exponential distribution. Journal of Statistical Computation and Simulation, **80**(2): 159-172. doi:10.1080/00949650802552402.

Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. Commun. Stat. Theory and Methods, **31**: 497-512.

Gupta, R. C., Gupta, P. I. and Gupta, R. D. (1998). Modeling failure time data by Lehmann alternatives. Commun. Stat. Theory Methods, **27**: 887–904.

Gupta, R. D. and Kundu, D. (2001). Exponentiated exponential family: an alternative to Gamma and Weibull distributions. Biometrical Journal, **43**: 117-130.

Maiti, S. S. and Pramanik, S. (2015). Odds generalized exponential - exponential distribution. Journal of Data Science, **13**: 733-754.

Hinkley, D. (1977). On quick choice of power transformation. Applied Statistics, **26**: 67-69.

Muhammad, M. (2016). Poisson-odd generalized exponential family of distributions: theory and applications. Hacettepe Journal of Mathematics and Statistics, **47**: 1-20.

Nadarajah, S. and Kotz, S. (2004). The beta gumbel distribution. Math. Probab. Eng., **10**: 323-332.

Nadarajah, S. and Kotz, S. (2005). The beta exponential distribution. Reliability Engineering and System Safety, **91**: 689-697.

Nasiru, S. (2018). A new generalization of Transformed-Transformer family of distributions. PhD Thesis, Pan African Institute of Basic Sciences, Technology and Innovation, Kenya.

Schwarz, G. (1978). Estimating the dimension of a model. The Annals of Statistics, **6**(2):461-464.

Singla, N., Jain, K. and Sharma, S. K. (2012). The Beta Generalized Weibull distribution: Properties and applications. Reliability Engineering and System Safety, **102**: 5 – 15.

Smith, R. L. and Naylor, J. C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. Applied Statistics, **36**: 358-369.

Tahir, M. H., Cordeiro, G. M., Alizadeh, M., Mansoor, M., Zubair, M. and Hamedani, G. G. (2015). The odd generalized exponential family of distributions with applications. Journal of Statistical Distribution and Applications.

Appendix A

R codes for computing the likelihood function, maximum likelihood estimates, AIC and BIC for the first data set.

```
datta<-c(0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73,
1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76,
1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76,
1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77,
1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89)

BOGE-HL
RRc<-function(theta,datta){
x<-datta
alpha<-theta[1]
lambda<-theta[2]
a<-theta[3]
b<-theta[4]
betta<- beta(a,b)
suv<-2*exp(-x)/(1+exp(-x))
odd<-(exp(x)-1)/2
g<-(2*exp(-x))/(1+exp(-x))^2
c12<-alpha*lambda*(2*betta)^(-1)
c22<-exp(x-lambda*odd)
c32<-(1-exp(-lambda*odd))^(alpha*a-1)
c42<-(1-(1-exp(-lambda*odd))^(alpha))^(b-1)
fddhl<-c12*c22*c32*c42
RRc<--sum(log(fddhl))
RRc
}
```

```
dd<-nlminb(start=c(0.001,0.1,0.01,.1),RRc,lower=c(0,0,0,0),
upper=c(Inf,Inf,Inf,Inf),datta=datta)
dd
===========================================================
```