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The Gumbel-Dagum distribution: a new member of the T-X family of distributions

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Abstract. A new member from the T-X family of distribution called the Gumbel-Dagum (GUD) distribution is introduced. This was realized by adopting the logit transformation of the cumulative distribution function of the Dagum random variable and the Gumbel distribution as the generator. Some mathematical properties of the new distribution are presented including the maximum likelihood .estimator of its parameters.

Keywords: T-X family, Logit function, Dagum distribution, Gumbel distribution, Maximum likelihood estimation.

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1. Introduction

Gumbel (1954) proposed Gumbel distribution. The Gumbel distribution is also known as the extreme value distribution of type I. The distribution is useful in predicting the chance that a natural disaster will occur. The distribution is perhaps the most widely applied statistical distribution for problems in engineering. Some of its recent application areas in engineering include flood frequency analysis, network engineering, nuclear engineering, offshore engineering, risk-based engineering, space engineering, software reliability engineering, structural engineering, and wind engineering, (Kotz and Nadarajah, 2000). In many applied areas like reliability, lifetime analysis, finance, and insurance, there is a clear need for extended forms of classical distributions, that is, new distributions which are more flexible to model real data in these areas.

The data in the areas of reliability, lifetime analysis, finance, and insurance can present a high degree of skewness and kurtosis and we can circumvent this problem by adding new parameters to the existing distributions (Gupta et al., 2016). A book by Kotz and Nadarajah (2000), listed over fifty applications which described this distribution, ranging from accelerated life testing through earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, wind speeds, and track race records.

On the other hand, Dagum distribution was proposed by Camilo Dagum (Dagum, 1977) for modeling personal income data as an alternative to the Pareto and the log-normal models. The Dagum distribution has been extensively used for fitting empirical income and wealth data, meteorological data, reliability and survival data. The distribution is also known as the inverse Burr XII distribution, in the actuarial literature (Domma, 2002). The Dagum distribution has an important characteristic in respect of the hazard rate function. The hazard function is monotonically decreasing, bathtub or upside-down bathtub shaped. Several researchers have used the distribution in different fields because of the different shapes of the hazard function. It has been studied from a reliability point of view and used to analyze survival data (Domma, et al. 2011; Domma and Condino, 2013). The

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class of weighted Dagum and related distribution was presented by Oluyede and Ye (2014), while the five parameter beta-Dagum distribution was proposed by Domma and Condino (2013).

In recent time there has been an increasing interest in defining new generators of univariate continuous distributions by adding shape parameter(s) to the baseline distribution. The introduction of parameters have been proved to be useful in investigating tail properties and improving the goodness-of-fit of the generated family. Some of the well-known generators are: beta-G (Eugene et al., 2002), beta-type distribution (Jones 2009), Kumaraswamy-G (Cordeiro and de Castro, 2011), T-normal (Alzaatreh et al. 2013b), transformed-transformer (T-X) family (Alzaatreh et al. 2013), T-R$\{Y\}$ family (Alzaatreh et al., 2014) and new Weibull-G (Tahir et al., 2016).

This study introduces a new member to the T -X family of distribution called the Gumbel-Dagum distribution. This is realized by adopting the logit transformation of the cumulative distribution function of the Dagum random variable with the Gumbel distribution as the generator. This flows from Alzaatreh et al. (2013) new method for generating new distributions.

Given a random variable X with probability density function (pdf), $f(x)$ and cumulative distribution function (cdf), $F(x)$, we utilize a random variable T defined on the interval $-\infty \leq < a < b \leq \infty$ with cdf $R(t)$ and pdf, $r(t)$, using a transformation $W[F(x)]$ of the cdf of X. Alzaatreh et al. (2013) defined a new class of distributions by the cdf of the form

$$G(x) = \int_{a}^{W[F(x)]]} r(t)dt \tag{1}$$

where $W[F(x)]$ satisfies the conditions

(i) $W[F(x)] \in [a, b]$,
(ii) $W[F(x)]$ is differentiable and monotonically non-decreasing
(iii) $W[F(x)] \to a$ as $x \to -\infty$ and $W[F(x)] \to b$ as $x \to \infty$.

Equation (1) is the "T-X distribution". Suppose $W[F(x)]$ is defined as the logit of the cdf, $F(X)$, i.e. $W[F(x)] = \ln\{F(x)/[1 - F(x)]\}$, which also corresponds to the quantile function of the standard logistic distribution. Then the cdf of the T-X distribution based on the logit function is given by

$$G(x) = \int_{-\infty}^{\ln(F(x)/(1-F(x)))} r(t)dt = R(\ln[F(x)/(1 - F(x))]). \tag{2}$$

The corresponding pdf, $f(x)$ of the T-X distribution is the derivative of $G(x)$ with respect to x, which gives

$$g(x) = \frac{h(x)}{F(x)} r(\ln[F(x)/(1 - F(x))]) \tag{3}$$

where $h(x) = \frac{f(x)}{[1-F(x)]}$ is the hazard function of the random variable X with cdf $F(x)$. Functionally, $R(t) = R(\ln\{F(x)/[1 - F(x)]\})$ implies $t = \ln\{F(x)/[1 - F(x)]\}$.

2. The Gumbel-Dagum distribution

Let T be a random variable that follows the Gumbel distribution with cdf and pdf given respectively by

$$R(t) = \exp\left[-\exp\left(-\left(\frac{t-\epsilon}{\alpha}\right)\right)\right] \tag{4}$$

and

$$r(t) = \frac{1}{\alpha}\exp\left(-\left(\frac{t-\epsilon}{\alpha}\right)\right)\exp\left[-\exp\left(-\left(\frac{t-\epsilon}{\alpha}\right)\right)\right] \tag{5}$$

$-\infty < t < \infty$, $\alpha > 0$, $-\infty < \epsilon < \infty$. α and ϵ are the scale and the location parameters, respectively (Johnson et al.,1994). Suppose also that X is a random variable that follows the Dagum distribution with cdf and pdf given respectively by

$$F(x) = (1 + \lambda x^{-\delta})^{-\beta} \tag{6}$$

and

$$f(x) = \beta\lambda\delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1} \tag{7}$$

for $x > 0$, where λ is a scale parameter, δ and β are shape parameters (Domma 2004). It is evident that

$$\ln(F(x)/[1 - F(x)]) = \ln((1 + \lambda x^{-\delta})^{\beta} - 1).$$

The cdf and the pdf of the GUD distribution using (2) are given respectively by

$$G(x) = 1 - \exp\left(-\exp(\epsilon/\alpha)\left((1 + \lambda x^{-\delta})^{\beta} - 1\right)^{1/\alpha}\right) \tag{8}$$

and

$$g(x) = \frac{\alpha\beta\delta\exp(\epsilon/\alpha)}{\alpha} x^{-\delta-1}\left((1 + \lambda x^{-\delta})^{\beta} - 1\right)^{-1-1/\alpha}\exp\left(-\exp(\epsilon/\alpha)\left((1 + \lambda x^{-\delta})^{\beta} - 1\right)^{-1/\alpha}\right) \tag{9}$$

$-\infty < \epsilon < \infty$, $x > 0$, $\alpha, \beta, \delta, \lambda > 0$. Equations (8) and (9) are the cdf and pdf of the 5-parameter GUD distribution with α, β, δ and ϵ as shape parameters and λ as scale parameter. For convenience and whenever it is necessary in the rest of the paper, we shall take $\theta = \exp(\epsilon/\alpha)$. Increasing the value of λ stretches the density curve and decreasing λ will shrink density curve. We observe that α and ϵ are scale and location parameters respectively in the Gumbel distribution but they become shape parameters in GUD distribution. The parameter λ is a scale parameter in the Dagum distribution and it also plays the role of a scale parameter in GUD distribution. The parameters λ is originally shape parameters in the Dagum distribution and also retain such property in the GUD distribution.

The GUD distribution $g(x)$ has different shapes including symmetric and bimodal shapes depending on different values of shape parameters. Some different shapes of the GUD distribution are shown in Figures 1 and 2.

3. Mathematical properties of the GUD distribution

The properties of the GUD distribution are examined and studied. These properties are vital when applying the distribution to real life data.

Survival function

The survival function of a positive random variable X defines the probability of survival at time x. The survival function associated with GUD distribution is

$$S(x) = 1 - G(x). \tag{10}$$

Substituting equation (8) into (10) gives

$$S(x) = 1 - \left(1 - \exp\left(-\theta[(1 + \lambda x^{-\delta})^{\beta} - 1]^{1/\alpha}\right)\right)$$

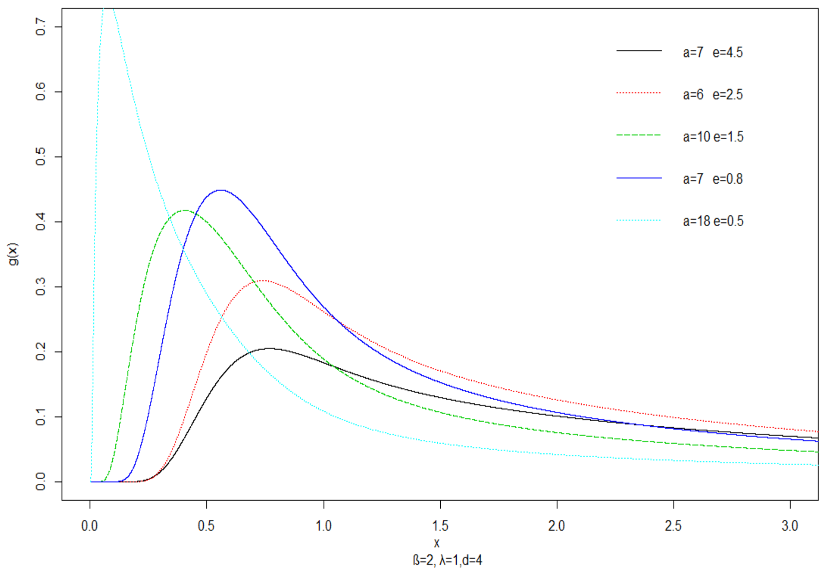

Figure 1.: Plots of the GUD distribution for selected parameter values.

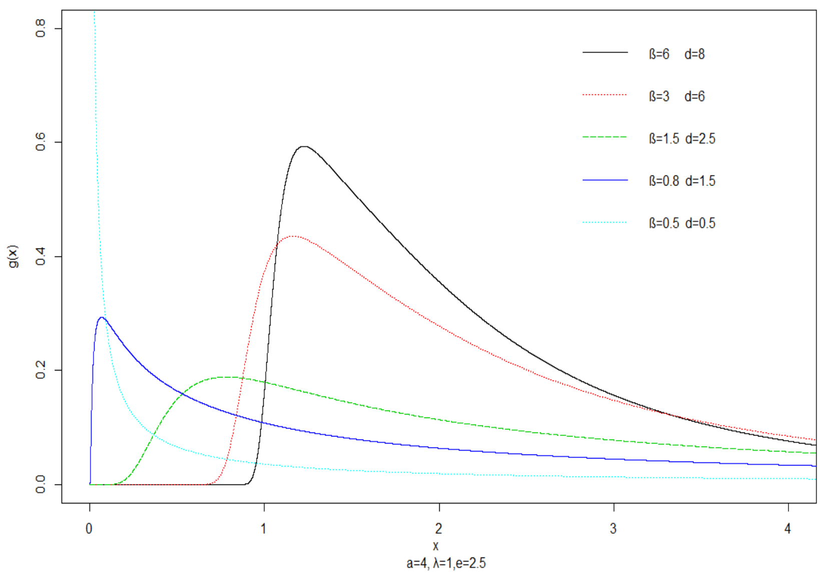

Figure 2.: Plots of the GUD distribution, $g(x)$, for selected parameter values.

$$S(x) = \exp\left(-\theta[(1+\lambda x^{-\delta})^{\beta} - 1]^{1/\alpha}\right). \tag{11}$$

Hazard function

The hazard function for the GUD distribution is given by

$$h(x) = \frac{\lambda\beta\delta\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{\beta-1}[(1+\lambda x^{-\delta})^{\beta}-1]^{-1-1/\alpha}\exp\left(-\theta[(1+\lambda x^{-\delta})^{\beta}-1]^{-1/\alpha}\right)}{\alpha\left(1-\exp\left(-\theta[(1+\lambda x^{-\delta})^{\beta}-1]^{1/\alpha}\right)\right)} \tag{12}$$

The hazard function in (12) exhibits different kind of shapes. The graph of the hazard function h(x) of the GUD is shown in Figure 3 and 4.

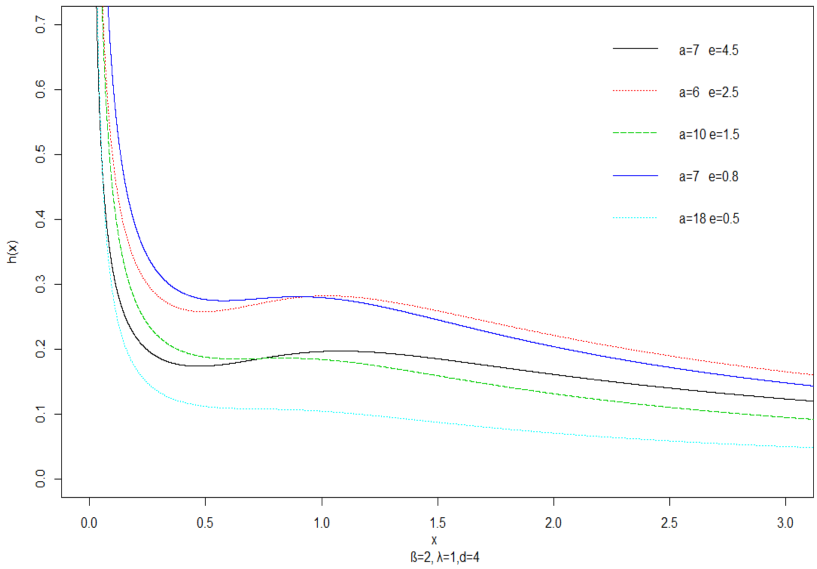

Figure 3.: Plots of the hazard function of GUD distribution for selected parameter values.

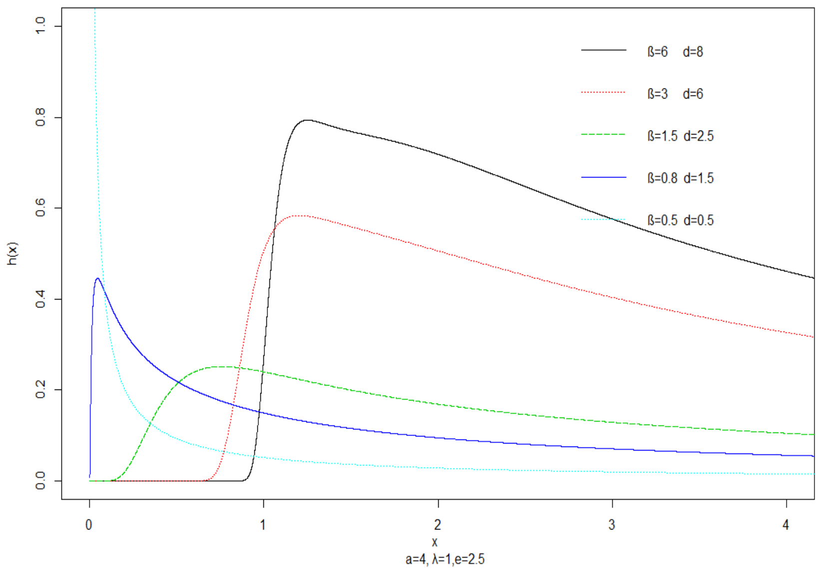

Figure 4.: Plots of the hazard function of GUD distribution for selected parameter values.

Quantile function

The quantile function of the T–X distribution is defined by $S(x) = \int r(t)dt$.

THEOREM 3.1 *The quantile function of the T–X distribution define by $S(x) = \int r(t)dt$ is given as*

$$Q_{T-X}(k) = Q_X\left(\exp(Q_T(k))/[1+\exp(Q_T(k))]\right), \qquad 0 < k < 1, \tag{13}$$

where $Q_{T(.)}$ is the quantile function of the random variable T with distribution function $R(t)$ and $Q_{X(.)}$ is the quantile function of the random variable X with distribution function $F(x)$.

Proof. Consider the equation $G(x) = k$. $G(x) = R[\ln(F(x)/(1-F(x)))] = k$. $x = Q_{T-X}(k) = F^{-1}\left(\exp(R^{-1}(k))/[1+\exp(R^{-1}(k))]\right)$ where $F^{-1}(.) = Q_X(.)$ and $R^{-1}(.) = Q_T(.)$ represent the quantile functions of the random variable X and T, respectively. Thus

$$Q_{T-X}(k) = Q_X\left(\exp(Q_T(k))/[1+\exp(Q_T(k))]\right).$$

■

Substituting $k = 0.5$ into (13) gives the median of the T-X distribution

$$Q_{T-X}(0.5) = Q_X\left(\exp(Q_T(0.5))/[1+\exp(Q_T(0.5))]\right).$$

Moments

The rth moment about zero of the random variable X of the GUD distribution, say μ_r^1, is given by

$$\mu_r^1 = \mathrm{E}(x^\tau) = \int_0^\infty x^\tau g(x)dx$$

$$\mu_r^1 = \int_0^\infty x^\tau \frac{\alpha\beta\delta\exp(\epsilon/\alpha)}{\alpha} x^{-\delta-1}\left((1+\lambda x^{-\delta})^\beta - 1\right)^{-1-1/\alpha}\exp\left(-\exp(\epsilon/\alpha)\left((1+\lambda x^{-\delta})^\beta-1\right)^{-1/\alpha}\right)dx$$

The cdf and the pdf of the GUD in equations (8) and (9) can be written in infinite series terms as

$$G(x) = \sum_{i=0}^{\infty}\frac{(-1)^i\mu^i}{i!}\left((1+\lambda x^{-\delta})^\beta - 1\right)^{-i/\alpha}. \tag{14}$$

and

$$g(x) = \sum_{i=0}^{\infty}\frac{(-1)^{i+1}\mu^i\beta\delta\lambda}{(i-1)!\alpha}x^{-\delta-1}(1+\lambda x^{-\delta})^{\beta-1}\left((1+\lambda x^{-\delta})^\beta - 1\right)^{-1-1/\alpha}. \tag{15}$$

Making use of (14) and (15), which are the infinite series terms of the cdf and pdf of GUD distributions, then the rth moment about zero of the GUD distribution can then be expressed as

$$\mu_\tau^1 = \sum_{i=0}^{\infty}\frac{(-1)^{i+1}\mu^i\beta\delta\lambda}{(i-1)!\alpha}\int_0^\infty x^{\tau-\delta-1}(1+\lambda x^{-\delta})^{\beta-1}\left((1+\lambda x^{-\delta})^\beta - 1\right)^{-1-i/\alpha}dx$$

$$\mu_\tau^1 = \sum_{i=0}^{\infty}\frac{(-1)^{i+1}\mu^i\beta\delta\lambda}{(i-1)!\alpha}B_1(x;\tau,\alpha,\lambda,\beta,\delta,i) \tag{16}$$

where

$$B_1(x;\tau,\alpha,\lambda,\beta,\delta,i) = \int_0^\infty x^{\tau-\delta-1}(1+\lambda x^{-\delta})^{\beta-1}\left((1+\lambda x^{-\delta})^\beta - 1\right)^{-1-i/\alpha}dx.$$

The first four moments about zero of GUD distribution are given as

$$\mu_1^1 = \sum_{i=0}^{\infty}\frac{(-1)^{i+1}\mu^i\beta\delta\lambda}{(i-1)!\alpha}B_1(x;1,\alpha,\lambda,\beta,\delta,i)$$

$$\mu_2^1 = \sum_{i=0}^{\infty} \frac{(-1)^{i+1}\mu^i \beta\delta\lambda}{(i-1)!\alpha} B_1(x; 2, \alpha, \lambda, \beta, \delta, i)$$

$$\mu_3^1 = \sum_{i=0}^{\infty} \frac{(-1)^{i+1}\mu^i \beta\delta\lambda}{(i-1)!\alpha} B_1(x; 3, \alpha, \lambda, \beta, \delta, i)$$

$$\mu_4^1 = \sum_{i=0}^{\infty} \frac{(-1)^{i+1}\mu^i \beta\delta\lambda}{(i-1)!\alpha} B_1(x; 4, \alpha, \lambda, \beta, \delta, i)$$

and

$$\mu = \frac{\beta\delta\lambda}{(\alpha} B_1(x; r, \alpha, \lambda, \beta, \delta, i).$$

The standard deviation of the Gumbel-Dagum distribution is given as

$$\sigma = \sqrt{\mu_2^1 - \mu^2}.$$

Skewness and Kurtosis

Coefficients of skewness (Sk) for the Gumbel-Dagum distribution can be obtained from the relations

$$Sk = \frac{\mu_3^1 - 3\mu\mu_2^1 + 2\mu^2}{(\mu_2^1 - \mu^2)^{3/2}} \tag{17}$$

and Coefficients of kurtosis (Ku) for the Gumbel Dagum distribution can be obtained from the relations

$$Ku = \frac{\mu_4^1 - 4\mu\mu_3^1 + 6\mu^2\mu_2^1 - 3\mu^4}{(\mu_2^1 - \mu^2)^2} \tag{18}$$

The two approaches will be used to examine the skewness and kurtosis of the GUD distribution. These approaches are (i) the measure of skewness and kurtosis based on moments defined in (17) and (18) respectively, and (ii) the measure of skewness and kurtosis based on quintiles. A regression analysis may also be used to establish the best relationship between the parameters. The quantile based approach is most useful when the quantile function of a distribution exists in a simple analytic expression or in closed form. A quantile based approach for evaluating skewness was proposed by Galton (1883) and the Kurtosis approach was proposed by Moor (1988). Galton's skewness is evaluated using the relation

$$S = \frac{Q(6/8) - 2Q(4/8) + Q(2/8)}{Q(6/8) - Q(2/8)}. \tag{19}$$

Moor's kurtosis is evaluated using the relation

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)} \tag{20}$$

where $Q(.)$ is the quantile function of the distribution under study.

4. Order statistics

Let the observations be ordered as $X_1, X_2, \cdots, X_n$, where X_1 denotes the minimum time to failure and X_n denotes the maximum time to failure. The trials are independent and identically distributed. The pdf of the kth order statistics from GUD distribution is

$$f_{k:n}(x) = \frac{n!g(x)}{(k-1)!(n-k)!}[G(x)]^{k-1}[1-G(x)]^{n-1} \tag{21}$$

Using the identity

$$(1-z)^{n-1} = \sum_{p=0}^{\infty}(-1)^p \binom{n-1}{p} z^p \tag{22}$$

$$f_{k:n}(x) = \frac{n!g(x)}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}(-1)^p \binom{n-k}{p} [G(x)]^{p+k-1} =$$

$$\frac{n!g(x)}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}(-1)^p \binom{n-k}{p} \left(1-\exp(-\theta[(1+\lambda x^{-\delta})^{\beta}-1]^{1/\alpha})\right)^{p+k-1}$$

$$= \frac{n!g(x)}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q} \binom{n-k}{p}\binom{p+k-1}{q} \left(1-\exp(-\theta[(1+\lambda x^{-\delta})^{\beta}-1]^{1/\alpha})\right)$$

$$= \frac{n!g(x)}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q} \binom{n-k}{p}\binom{p+k-1}{q} \frac{\lambda\beta\delta\theta}{\alpha}x^{-\delta-1}(1+\lambda x^{-\delta})^{\beta-1}\times$$

$$[(1+\lambda x^{-\delta})^{\beta}-1]^{-1-1/\alpha}\exp(-\theta[(1+\lambda x^{-\delta})^{\beta}-1]^{-1/\alpha})\left(1-\exp(-\theta[(1+\lambda x^{-\delta})^{\beta}-1]^{1/\alpha})\right) \tag{23}$$

5. Estimation of the Gumbel-Dagum distribution parameters

Let $x_1, x_2, \cdots, x_n$ denote a random sample drawn from the Gumbel-Dagum distribution with parameters θ, α, β, δ and λ defined in (9). The likelihood function $f(x_1, x_2, \cdots, x_n; \theta, \alpha, \beta, \delta, \lambda)$ is defined to be the joint density of the random variables $x_1, x_2, \cdots, x_n$. That is

$$L(\mathbf{x};\theta,\alpha,\beta,\delta,\lambda) = \prod_{i=1}^{n} f(x_i;\theta,\alpha,\beta,\delta,\lambda) \tag{24}$$

The likelihood of the Gumbel-Dagum distribution function is

$$L(\mathbf{x};\theta,\alpha,\beta,\delta,\lambda) = \prod_{i=1}^{n}\frac{\beta\delta\lambda\theta}{\alpha}x^{-\delta-1}(1+\lambda x^{-\delta})^{\beta-1}\left[(1+\lambda x^{-\delta})^{\beta}-1\right]^{-1-1/\alpha}\times$$

$$\exp\left(-\theta[(1+\lambda x^{-\delta})^{\beta}-1]^{-1/\alpha}\right) \tag{25}$$

and

$$\log L(\mathbf{x};\theta,\alpha,\beta,\delta,\lambda)=n[\log\beta+\log\lambda+\log\delta+\log\theta-\log\alpha]-(\delta+1)\sum_{i=1}^{n}\log x_i+$$

$$(\beta-1)\sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta})-(1+1/\alpha)\sum_{i=1}^{n}\log\left[(1+\lambda x^{-\delta})^{\beta}-1\right]-\theta\sum_{i=1}^{n}[(1+\lambda x^{-\delta})^{\beta}-1]^{-1/\alpha}. \quad (26)$$

Let $\Theta=(\alpha,\beta,\lambda,\delta,\theta)^T$ be the unknown parameter vector. The score vector which is the gradient of the log-likelihood function with respect to the parameters being estimated is given by

$$\bigcup(\Theta)=\left(\frac{\partial L}{\partial\alpha},\frac{\partial L}{\partial\beta},\frac{\partial L}{\partial\lambda},\frac{\partial L}{\partial\delta},\frac{\partial L}{\partial\theta}\right)$$

where $\frac{\partial L}{\partial\alpha}$, $\frac{\partial L}{\partial\beta}$, $\frac{\partial L}{\partial\lambda}$, $\frac{\partial L}{\partial\delta}$ and $\frac{\partial L}{\partial\theta}$ are the partial derivatives of the log-likelihood function with respect to each parameters and are given by

$$\frac{\partial L}{\partial\alpha}=-\frac{n}{\alpha}-\frac{n\delta}{\alpha}+\frac{1}{\alpha^2}\sum_{i=1}^{n}\log[(1+\lambda x^{-\delta})^{\beta}-1]-\frac{\theta}{\alpha^2}\sum_{i=1}^{n}\log[(1+\lambda x^{-\delta})^{\beta}-1][(1+\lambda x^{-\delta})^{\beta}-1]^{-1/\alpha} \quad (27)$$

$$\frac{\partial L}{\partial\beta}=-\frac{n}{\beta}+\sum_{i=1}^{n}\log[(1+\lambda x_i^{-\delta})^{\beta}]+\left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{n}\frac{\log[(1+\lambda x_i^{-\delta})][(1+\lambda x_i^{-\delta})^{\beta}]}{[(1+\lambda x_i^{-\delta})^{\beta}-1]}$$

$$+\frac{\theta}{\alpha}\sum_{i=1}^{n}\log[(1+\lambda x_i^{-\delta})][(1+\lambda x^{-\delta})^{\beta}][(1+\lambda x_i^{-\delta})^{\beta}-1]^{-1-1/\alpha} \quad (28)$$

$$\frac{\partial L}{\partial\theta}=-\frac{n}{\theta}-\sum_{i=1}^{n}\log[(1+\lambda x_i^{-\delta})^{\beta}-1]^{-1/\alpha} \quad (29)$$

$$\frac{\partial L}{\partial\lambda}=-\frac{n}{\lambda}-(\beta-1)x^{-\delta}\sum_{i=1}^{n}\log[(1+\lambda x_i^{-\delta})^{\beta}]-\left(1+\frac{1}{\alpha}\right)x^{-2\delta}\sum_{i=1}^{n}\log[(1+\lambda x_i^{-\delta})^{\beta}]\times$$

$$[(1+\lambda x_i^{-\delta})^{\beta}-1]-\theta x^{-2\delta}\sum_{i=1}^{n}\log[(1+\lambda x_i^{-\delta})^{\beta}][(1+\lambda x_i^{-\delta})^{\beta}-1]^{-1/\alpha} \quad (30)$$

$$\frac{\partial L}{\partial\delta}=-\frac{n}{\delta}+\sum_{i=1}^{n}\log x_i-(\beta-1)\delta\sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta-1})+\left(1+\frac{1}{\alpha}\right)\delta^2\sum_{i=1}^{n}\log[(1+\lambda x_i^{-\delta})^{\beta}]\times$$

$$[(1+\lambda x_i^{-\delta})^{\beta}-1]+\theta\delta^2\sum_{i=1}^{n}\log[(1+\lambda x_i^{-\delta-1})^{\beta}][(1+\lambda x_i^{-\delta-1})^{\beta}-1]^{-1/\alpha} \quad (31)$$

To obtain the maximum likelihood estimate of Θ, we solve the non-linear systems of equations $\bigcup(\Theta) = \mathbf{0}$ numerically since the resulting system of equations are not in closed form.

6. Application

In this section, we present an example to illustrate the flexibility of the GUD distribution. We demonstrate the fit of the model of GUD by means of real data set to show its potentiality. In order to estimate the parameters of this model, we adopted the maximum likelihood method and provide standard errors of the estimators. The data set consists of a real data obtained from Nicholas and Padgett (2006), which are the breaking stress of carbon fibers of 50 mm Length (GPa). The data is unimodal and approximately symmetric (Skewness = -0.1315 and kurtosis = 0.2231).

Table 1.: Breaking stress of carbon fibers of 50 mm Length (n=66)

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12,
2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85,
2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31,
3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Table 2.: MLE's standard errors (in parenthesis) and AIC values for the Breaking stress of carbon fibers data

Distribution Parameters	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\delta}$	$\hat{\lambda}$	Log-likelihood	AIC
Gumbel–Dagum (GUD)	2.3198 (7.4381)	0.4665 (0.4135)	0.035 (0.0589)	8.2186 (25.788)	9.5827 (28.278)	-85.3836	180.7671

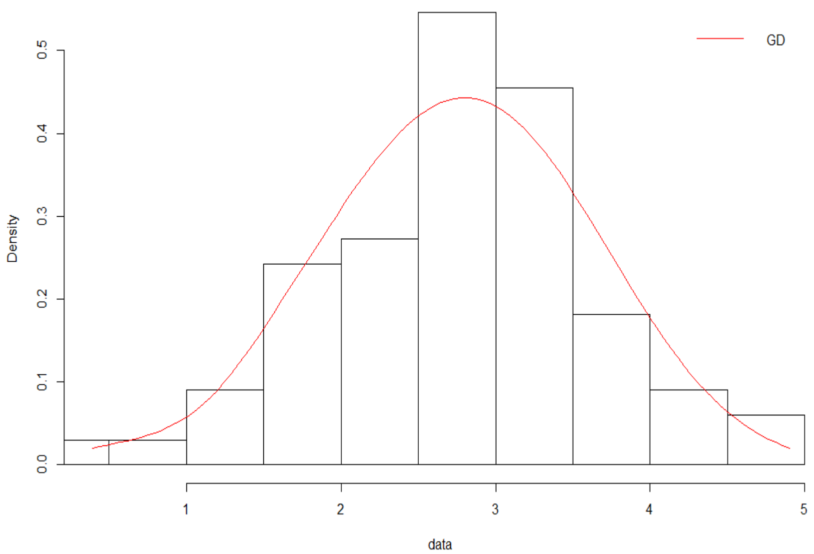

Figure 5.: Plot of the observed histogram and estimated pdf's for GUD distribution for the data set.

The results obtained suggest that the proposed distribution fits the data set. The goodness-of-fit is shown in Figure 5.

7. Conclusion

In this paper, we derived a new member from the T-X family of distributions called the Gumbel-Dagum distribution. Some properties of the new distribution such as hazard rate, quantile function, moments and mode are obtained. The maximum likelihood method is employed to estimate the model parameters. The new distribution fits the data set considered in the study adequately.

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Appendix

```
#===================================================
#R programme to plot the #GUDAGUM density and hazard functions
#===================================================
#GUDAGUM density and cumulative distribution functions
#a=alpha,b=beta, c=lambda, d=dell,e=epsilon
pGUDAGUM<-function(x,a,b,c,d,e)  exp(-exp(e/a)*((1+c*x^-d)^b-1)^(1/a))
dGUDAGUM<-function(x,a,b,c,d,e) (c*b*d*exp(e/a)/a)*x^(-d-1)* (1+c*x^-d)^(b-1)*((1+c*x^-d)^b-
1)^(-1+(1/a))*  exp(-exp(e/a)*((1+c*x^-d)^b-1)^(1/a))

#===================================================
```

Appendix 1: The R programme to plot the Gumbel-Dagum density of figure 1.

```
#GUDAGUM density plot 1
x<-seq(0.001,5,0.0001)
y1<-dGUDAGUM(x,7,2,1,4,4.5)
y2<-dGUDAGUM(x,6,2,1,4,2.5)
y3<-dGUDAGUM(x,10,2,1,4,1.5)
y4<-dGUDAGUM(x,7,2,1,4,0.8)
y5<-dGUDAGUM(x,18,2,1,4,0.5)
plot(c(0,3),c(0,0.7),type="n",ylab="g(x)",xlab="x, β=2, λ=1,δ=4")
lines(x,y1,col=1,lty=1)
lines(x,y2,col=2,lty=3)
lines(x,y3,col=3,lty=5)
lines(x,y4,col=4,lty=7)
lines(x,y5,col=5,lty=9)
legend("topright",c(
  "α=7   ε=4.5",
  "α=6   ε=2.5",
  "α=10 ε=1.5",
  "α=7   ε=0.8",
  "α=18 ε=0.5"),
  col=1:5, bty="n",lty=c(1,3,5,7,9))
#α,ε,β,δ,λ
```

#==

Appendix 2: The R programme to plot the Gumbel-Dagum density of figure 2

```
#GUDAGUM density plot 2

x<-seq(0.001,5,0.0001)
y1<-dGUDAGUM(x,4,6,1,8,2.5)
y2<-dGUDAGUM(x,4,3,1,6,2.5)
y3<-dGUDAGUM(x,4,1.5,1,2.5,2.5)
y4<-dGUDAGUM(x,4,0.8,1,1.5,2.5)
y5<-dGUDAGUM(x,4,0.5,1,0.5,2.5)
plot(c(0,4),c(0,0.8),type="n",ylab="g(x)",xlab="x

   α=4, λ=1,ε=2.5")

lines(x,y1,col=1,lty=1)

lines(x,y2,col=2,lty=3)

lines(x,y3,col=3,lty=5)

lines(x,y4,col=4,lty=7)

lines(x,y5,col=5,lty=9)

legend("topright",c("β=6    δ=8",  "β=3     δ=6",  "β=1.5  δ=2.5",  "β=0.8  δ=1.5",  "β=0.5  δ=0.5"),

 col=1:5, bty="n",lty=c(1,3,5,7,9))
```

#===

Appendix 3: The R programme to plot the Gumbel-Dagum hazard function of figure 3.

```
#GUDAGUM hazard plot 1
x<-seq(0.001,5,0.0001)

y1<-dGUDAGUM(x,7,0.2,1,4,4.5);z1<-pGUDAGUM(x,7,0.2,1,4,4.5)

y2<-dGUDAGUM(x,6,0.2,1,4,2.5);z2<-pGUDAGUM(x,6,0.2,1,4,2.5)

y3<-dGUDAGUM(x,10,0.2,1,4,1.5);z3<-pGUDAGUM(x,10,0.2,1,4,1.5)

y4<-dGUDAGUM(x,7,0.2,1,4,0.8);z4<-pGUDAGUM(x,7,0.2,1,4,0.8)

y5<-dGUDAGUM(x,18,0.2,1,4,0.5);z5<-pGUDAGUM(x,18,0.2,1,4,0.5)

plot(c(0,3),c(0,0.7),type="n",ylab="h(x)",xlab="x  β=2, λ=1,δ=4")

lines(x,y1/(1-z1),col=1,lty=1)

lines(x,y2/(1-z2),col=2,lty=3)

lines(x,y3/(1-z3),col=3,lty=5)

lines(x,y4/(1-z4),col=4,lty=7)

lines(x,y5/(1-z5),col=5,lty=9)

legend("topright",c("α=7   ε=4.5",  "α=6   ε=2.5",  "α=10 ε=1.5",  "α=7   ε=0.8",  "α=18 ε=0.5"),

 col=1:5, bty="n",lty=c(1,3,5,7,9))
```

```
#============================================================
Appendix 4: The R programme to plot the Gumbel-Dagum hazard function of figure 4.

#GUDAGUM hazard plot 2

x<-seq(0.001,5,0.0001)
y1<-dGUDAGUM(x,4,6,1,8,2) ;z1<-pGUDAGUM(x,4,6,1,8,2)
y2<-dGUDAGUM(x,4,3,1,6,2) ;z2<-pGUDAGUM(x,4,3,1,6,2)
y3<-dGUDAGUM(x,4,1.5,1,2.5,2) ;z3<-pGUDAGUM(x,4,1.5,1,2.5,2)
y4<-dGUDAGUM(x,4,0.8,1,1.5,2) ;z4<-pGUDAGUM(x,4,0.8,1,1.5,2)
y5<-dGUDAGUM(x,4,0.5,1,0.5,2); z5<-pGUDAGUM(x,4,0.5,1,0.5,2)
plot(c(0,4),c(0,1),type="n",ylab="h(x)",xlab="x  α=4, λ=1,ε=2.5")
lines(x,y1/(1-z1),col=1,lty=1)
lines(x,y2/(1-z2),col=2,lty=3)
lines(x,y3/(1-z3),col=3,lty=5)
lines(x,y4/(1-z4),col=4,lty=7)
lines(x,y5/(1-z5),col=5,lty=9)
legend("topright",c("β=6     δ=8", "β=3     δ=6", "β=1.5  δ=2.5", "β=0.8  δ=1.5", "β=0.5  δ=0.5"),
  col=1:5, bty="n",lty=c(1,3,5,7,9))

#============================================================
Appendix 5: The R programme to plot the Gumbel-Dagum fitted density of figure 5.
R codes
library(fitdistrplus)
#==============================================================

#Gumbel - Dagum Distribution
#==============================================================

pGD<-function(q,a,b,c,d,e) exp(-c*((1+e*q^-d)^b-1)^(-1/a))

dGD<-function(x,a,b,c,d,e) ((e*b*d*c)/a)*x^(-d-1)*(1+e*x^-d)^(b-1)*

((1+e*x^-d)^b-1)^(-1-1/a)*exp(-c*((1+e*x^-d)^b-1)^(-1/a))

#=====================================================================

x3=c(0.39,0.85,1.08,1.25,1.47,1.57,1.61,1.61,1.69,1.80,1.84,1.87,1.89Zxc,2.03,2.03,2.05,2.12,2.35,2.41,2.43,2.48,
2.50,2.53,2.55,2.55,2.56,2.59,2.67,2.73,2.74,2.79,2.81,2.82,2.85,2.87,2.88,2.93,2.95,2.96,2.97,3.09,3.11,3.11,3.15,
3.15,3.19,3.22,3.22,3.27,3.28,3.31,3.31,3.33,3.39,3.39,3.56,3.60,3.65,3.68,3.70,3.75,4.20,4.38,4.42,4.70,4.90)
#========================================================

y2<-fitdist(x3,"GD", start=list(a=7.5,b=2.5,c=2.5,d=2.5,e=0.4))

summary(y2)

gofstat(y2)

denscomp(y2, main="")

ks.test.imp(x3,"pGD", a=y1$estimate[1],b=y1$estimate[2],c=y1$estimate[3],d=y1$estimate[4],e=y1$estimate[5])
```