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Lomax-Cauchy{Uniform} Distribution: Properties and Application to Exceedances of Flood Peaks of Wheaton River

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Abstract. In this article, we propose a new distribution called Lomax-Cauchy {Uniform} distribution (LCU) distribution using Cauchy distribution as the baseline distribution. Statistical properties such as mode, median, density of order statistic, quantile fnction, skewness and kurtosis of the new Lomax-Cauchy {Uniform} distribution. We investigate the shapes of the density, hazard rate function, cummulative hazard function,linear representation of LCU distribution and asymptotic properties of the new Lomax-Cauchy {Uniform} distribution using simulation when n=20, 100, 500 and 1000. The measure of variation of uncertainty of random variable of LCUD is also determined. Maximum Likelihood estimation method is adopted in estimating the parameters. Lastly, the importance of the new LCU distribution is illustrated by comparing the proposed distribution with some existing distributions. The results show that the LCUD model provides consistently better fits than other competing models for the exceedances of flood peaks of Wheaton River data.

Keywords: Cauchy, flood peaks, Lomax, Lomax-Cauchy {Uniform} distribution, MLE, quantile function.

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1. Introduction

Recently, global warming has caused high rainfall in many countries. Rivers overflowed their banks and flooded most communities. Several flood warnings are in effect for some rivers. It is necessary to study flood peaks of rivers in order to give accurate predictions of flood warnings in affected areas. The Arctic and sub-Arctic regions are experiencing greater climate warming than other parts of the world (Intergovernmental Panel on Climate Change, 2007). In the past decades, studies have shown that temperatures in these areas have increased at twice the rate of the rest of the planet causing atmospheric warming in Yukon Territory (Arctic Climate Impact Assessment, 2005; Lemmen et al., 2008; Hare at al., 2008). Most of the existing distributions have not fitted the flood peaks of the Wheaton river, which is experiencing major climate change. Alshawarbeh et al. (2013) fitted the flood peaks of Wheaton river with Beta-Cauchy distribution and looks like a good fit. But there could be distributions that can explain the variability in the flood peaks of the river better.

The Cauchy distribution, named after Augustin Cauchy, is a simple family of distributions for which the expected value does not exist. The Cauchy distribution has been used in many applications such as mechanical and electrical theory, physical anthropology, measurement problems, risk and financial analysis. It was also used to model the points of impact of a fixed straight line of particles emitted from a point source (Johnson et al. 1994). In Physics, it is called a Lorenzian distribution, where it is the distribution of the energy of an unstable state in quantum mechanics.

The Cauchy distribution finite moment of order greater than or equal to one does not exist except for fractional absolute moments (Riley, et al., 2006). The central limit theorem does not hold for the limiting distribution of the mean of a random sample from a Cauchy distribution. Due to this feature of the distribution, some authors consider the Cauchy distribution as a pathological case (Krishnamoorthy, 2006). The Cauchy distribution is an extreme case distribution and serves as counter examples for some well accepted

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results and concepts in statistics. (Ohakwe and Osu, 2011).

On the other hand, the Lomax distribution, also known as Pareto type-II distribution, is one of the important continuous distributions with a heavy tail defined by one shape and one scale parameters. The Lomax distribution was first used in Lomax (1954) to analyze business failure data. After that, researchers turned to using the Lomax distribution extensively in applications in the different fields of sciences; this included but was not limited to modeling business records by Atkinson and Harrison (1978), reliability and life testing studies by Hassan and Al-Ghamdi (2009). Bryson (1974) suggested the use of the Lomax distribution as an alternative to exponential, gamma and Weibull distributions. Ogunsanya et al. (2019) proposed Odd Lomax-Exponential Distribution and applied it to daily arrival of patients per hour at the University of Lagos Teaching Hospital for April 2017. Other applications of the Lomax distribution can be found in modeling heavy tailed data in wealth, income, business and biological sciences.

Some of the known techniques to generalize distributions in the recent decades are introduced by Marshall and Olkin (1997), Eugene et al. (2002), Shaw and Buckley (2009), Alzaatreh et al. (2013), and Alzaghal et al.(2013). Eugene et al. (2002) introduced the beta-generated family of distributions using the beta as the baseline distribution. Based on the beta-generated family, Alshawarbeh et al. (2013) proposed the beta-Cauchy distribution. The beta-generated family was extended by Alzaatreh et al. (2013) to the T-R(W) family. Aljarrah et al. (2014) considered the function W(.) to be the quantile function of a random variable Y and defined the T-R$\{Y\}$ family. Many families of these generalized distributions have appeared in the literature. Alzaatreh et al. (2014), and Alzaatreh and Ghosh (2015) studied the T-gamma and the T-normal families. Almheidat et al. (2015) studied the T-Weibull family.

In this article, a member of the family of generalized Cauchy distribution, which was first studied by Alzaatreh et al. (2016), called the Lomax-Cauchy distribution using standard uniform distribution quantile function is proposed and studied. The study includes moments, estimation, simulation and applications. This proposed Lomax-Cauchyuniform distribution is used to model flood peaks in Yukon by examining the highest stage reached during a particular flood at a given point on a Wheaton river. The remaining part of this article is presented as follows:

2. Materials and method

2.1 *Derivation of Lomax-Cauchy$\{Uniform\}$ distribution*

In the T-R$\{Y\}$ framework, the random variable T is a 'transformer' that is used to 'transform' the random variable R into a new family of generalized distributions of R. Note that T, R and Y are combined together to form a new random variable X. The cumulative density function (CDF) of X as defined by Aljarrah et al. (2014) is given by

$$F_X(x) = \int_a^{Q_Y[F_R(x)]} f_T(t)dt = F_T\{Q_Y[F_R(x)]\} \tag{1}$$

and the corresponding probability density function (PDF) is given by

$$f_X(x) = f_R(x)\frac{f_T\{Q_Y[F_R(x)]\}}{f_Y\{Q_Y[F_R(x)]\}}. \tag{2}$$

Alternatively, the PDF in (2) can be written as

$$f_X(x) = f_T\{Q_Y[F_R(x)]\} \times f_Y\{Q_Y^{'}[F_R(x)]\} \times f_R(x). \tag{3}$$

Let R be a random variable that follows the Cauchy distribution with CDF given by

$$F_R(x) = \frac{1}{2} + \frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right). \tag{4}$$

Let Y be a random variable that follows uniform distribution in the bounded interval [0,1] with a quantile function given by

$$Q_Y(x) = x \tag{5}$$

By using the quantile function of the uniform distribution given in (5), Alzaatreh et al. (2016) derived the corresponding CDF to (1) as

$$F_X(x) = F_T[F_R(x)] \tag{6}$$

and the corresponding PDF to (6) is given by

$$f_X(x) = f_C(x) \times f_T[F_C(x)] \tag{7}$$

where $F_C(x)$ and $f_C(x)$ are the CDF of Cauchy distribution, that is, $F_C(x) = F_R(x)$ and $f_T(x)$ is the PDF of Lomax distribution given by

$$f_T(x) = \frac{\lambda k}{(1+\lambda x)^{k+1}} \tag{8}$$

and the CDF of Lomax distribution is given by

$$F_T(x) = 1 - (1+\lambda x)^{-k} \tag{9}$$

Put equation (4) into (9) to have

$$F_T[F_R(x)] = 1 - (1+\lambda F_R(x))^{-k} \tag{10}$$

Now, put (10) into (6) to have the CDF of the proposed Lomax-Cauchy$\{Uniform\}$ (LCU) distribution given by

$$F_X(x) = 1 - \left\{1+\lambda\left[\frac{1}{2}+\frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]\right\}^{-k} \tag{11}$$

and the corresponding PDF to (11) is given by

$$f_X(x) = \frac{\lambda k}{\alpha\pi}\left[\left(\frac{x-\theta}{\alpha}\right)^2+1\right]^{-1}\left\{1+\lambda\left[\frac{1}{2}+\frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]\right\}^{-k-1}, \ \alpha,k,\lambda>0, \theta\geq 0, x\geq 0 \tag{12}$$

where k is a shape parameter, θ is a location parameter, λ and α are scale parameters.

2.2 *Properties of LCU distribution*

The structural properties of LCU distribution will be investigated in details in this section. For ease, a random variable X with PDF $f_X(x)$ in (12) is said to follow the lomax-Cauchy$\{uniform\}$ distribution and is denoted by LCU$(\alpha, k, \lambda, \theta)$.

2.2.1 Survival function

The survival function of LCU$(\alpha, k, \lambda, \theta)$ is given by

$$S_X(x) = \left\{1+\lambda\left[\frac{1}{2}+\frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]\right\}^{-k} \tag{13}$$

The function in (13) is a function that gives the probability that an item of interest will survive beyond any given specified time $x \in X$, also known as the reliability function.

2.2.2 *Hazard function*

The hazard function of the LCU$(\alpha, k, \lambda, \theta)$ is given by

$$h_X(x) = \frac{\lambda k}{\alpha\pi}\left[\left(\frac{x-\theta}{\alpha}\right)^2 + 1\right]^{-1}\left\{1 + \lambda\left[\frac{1}{2} + \frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]\right\}^{-1} \tag{14}$$

The function in (14) of random variables X that follows LCU distribution is called the hazard rate of LCU distribution, also known as the failure rate or force of mortality.

2.2.3 *Cumulative hazard function*

The cumulative hazard function, $H_X(x)$ of a continuous random variable X that follows the LCU distribution is derived from this definition

$$H_X(x) = -log[S_X(x)] \tag{15}$$

Put equation (13) into (15) to have

$$H_X(x) = klog\left(\left\{1 + \lambda\left[\frac{1}{2} + \frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]\right\}\right) \tag{16}$$

Equation (16) defines the probability of failure at time x given survival until time $x\,\forall\,x \in X$.

2.2.4 *Quantile function, median, skewness, kurtosis and mode*

LEMMA 2.1 *Let Y be a random variable that follows a Lomax distribution with parameters k and λ, then the quantile function of LCU$(\alpha, k, \lambda, \theta)$ is given by $Q_X(p) = \theta + \alpha tan\left[\pi\left(Q_Y(p) - 0.5\right)\right]$.*

Proof. If Y is a Lomax random variable with parameters λ and k, then the random variable

$$X = \theta + \alpha tan\left[\pi\left(Y - 0.5\right)\right] \tag{17}$$

follows a LCU$(\alpha, k, \lambda, \theta)$ in equation (12. By using equation(17), the LCU quantile function can be computed from the lomax quantile function as

$$x = Q_X(p) = \theta + \alpha tan\left[\pi\left(Q_Y(p) - 0.5\right)\right] \tag{18}$$

where $Q_Y(p)$ is the quantile function of the lomax distribution with parameters λ and k. We can compute $Q_Y(p)$ by using R code. ■

THEOREM 2.2 *Let X be a random variable that follows a LCU$(\alpha, k, \lambda, \theta)$, the quantile function of $X$is given by $Q_X(p) = \theta + \alpha tan\left[\pi\left(\left[\frac{1}{\lambda}(1-p)^{-1/k} - \frac{1}{\lambda}\right] - 0.5\right)\right]$*

Proof. If Y follows a lomax distribution with parameters λ and k, then its quantile function is given by

$$Q_T(p) = \frac{1}{\lambda}(1-p)^{-\frac{1}{k}} - \frac{1}{\lambda} \tag{19}$$

Put equation (19) into (18) to arrive at the quantile function of LCU$(\alpha, k, \lambda, \theta)$ and it is given by.

$$Q_X(p) = \theta + \alpha tan\left[\pi\left(\left[\frac{1}{\lambda}(1-p)^{-1/k} - \frac{1}{\lambda}\right] - 0.5\right)\right] \tag{20}$$

■

Median

The median and other measures of partition for LCU distribution can be derived from the quantile function in equation (20).

The median of LCU$(\alpha, k, \lambda, \theta)$ is given by

$$M_e = Q_X(0.5) = \theta + \alpha tan\left[\pi\left(\left[\frac{0.5^{-1/k}}{\lambda} - \frac{1}{\lambda}\right] - 0.5\right)\right] \tag{21}$$

2.2.5 *Skewness and kurtosis*

It is often known that the coefficient of skewness and kurtosis are derived from the moment of a distribution. But an alternative measures for skewness and kurtosis is based on quantile functions, which are more appropriate for T-Cauchy distribution classes. The measure of skewness S and kurtosis K defined by Galton (1983) and Moors (1988) are based on quantile functions and they are respectively defined as

$$S = \frac{Q(6/8) - 2Q(4/8) + Q(2/8)}{Q(6/8) - Q(2/8)} \tag{22}$$

and

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)} \tag{23}$$

THEOREM 2.3 *The skewness and kurtosis of LCU distribution does not depend on parameters α and θ*

Proof. Let

$$\phi(p) = \pi\left\{\frac{1}{\lambda}\left[(1-p)^{-1/k} - 1\right] - 0.5\right\} \tag{24}$$

Put equation (24) in (20) to have

$$Q_X(p) = \theta + \alpha tan[\phi(p)] \tag{25}$$

Put equation (25) in (22) to have

$$S = \frac{\theta + \alpha tan[\phi(6/8)] - 2\theta - \alpha tan[\phi(4/8)] + \theta + \alpha tan[\phi(2/8)]}{\theta + \alpha tan[\phi(6/8)] - \theta - \alpha tan[\phi(2/8)]} \tag{26}$$

Factorise equation (26) by factoring out α, since all the terms contains α to have

$$S = \frac{tan[\phi(6/8)] - tan[\phi(4/8)] + tan[\phi(2/8)]}{tan[\phi(6/8)] - tan[\phi(2/8)]} \tag{27}$$

Thus, $\phi(.)$ does not contain α and θ.

Also, put equation (25) in (23) to have

$$K = \frac{\theta + \alpha tan[\phi(7/8)] - \theta - \alpha tan[\phi(5/8)] + \theta + \alpha tan[\phi(3/8)] - \theta - \alpha tan[\phi(1/8)]}{\theta + \alpha tan[\phi(6/8)] - \theta - \alpha tan[\phi(2/8)]} \tag{28}$$

Factorise equation (28) by factoring out α, since all the terms contains α to have

$$K = \frac{tan[\phi(7/8)] - tan[\phi(5/8)] + tan[\phi(3/8)] - tan[\phi(1/8)]}{tan[\phi(6/8)] - tan[\phi(2/8)]} \tag{29}$$

Thus, $\phi(.)$ in equation (29) does not contain α and θ. Equations (27) and (29) complete the proof. ■

For some varying values of λ and k, the values of Galton's skewness and Moors' kurtosis are computed and presented in Table 1.

Table 1: Galton's skewness and Moors' kurtosis for some values of k, λ (for any α and θ)

k	λ	Skewness	Kurtosis	k	λ	Skewness	Kurtosis
0.5	0.5	-1.0000	0.2129	20.0	0.5	0.2321	0.9809
0.5	1.0	-1.0000	-0.1960	20.0	1.0	0.2418	0.9617
0.5	5.0	20.9261	-9.5666	20.0	5.0	0.2661	1.0117
0.5	10.0	0.2971	-2.9166	20.0	10.0	0.2702	1.0229
0.5	15.0	5.8974	4.3920	20.0	15.0	0.2716	1.0269
0.5	20.0	1.2445	0.1741	20.0	20.0	0.2724	1.0289
1.0	0.5	-1.0000	-0.0276	25.0	0.5	0.2309	0.9559
1.0	1.0	-1.0000	-36.1911	25.0	1.0	0.2440	0.9622
1.0	5.0	0.8476	0.0596	25.0	5.0	0.2652	1.0092
1.0	10.0	0.4855	-8.7244	25.0	10.0	0.2685	1.0183
1.0	15.0	0.4563	2.7201	25.0	15.0	0.2697	1.0215
1.0	20.0	0.4545	2.0522	25.0	20.0	0.2703	1.0231
5.0	0.5	0.9255	-0.0999	30.0	0.5	0.2321	0.9477
5.0	1.0	0.3337	2.1510	30.0	1.0	0.2460	0.9645
5.0	5.0	0.2819	1.0659	30.0	5.0	0.2646	1.0076
5.0	10.0	0.2948	1.0943	30.0	10.0	0.2674	1.0152
5.0	15.0	0.3002	1.1092	30.0	15.0	0.2684	1.0179
5.0	20.0	0.3030	1.1179	30.0	20.0	0.2689	1.0192
10.0	0.5	0.2961	1.5878	35.0	0.5	0.2339	0.9457
10.0	1.0	0.2446	1.0211	35.0	1.0	0.2477	0.9671
10.0	5.0	0.2707	1.0259	35.0	5.0	0.2642	1.0065
10.0	10.0	0.2785	1.0460	35.0	10.0	0.2666	1.0130
10.0	15.0	0.2813	1.0540	35.0	15.0	0.2675	1.0153
10.0	20.0	0.2827	1.0582	35.0	20.0	0.2679	1.0165
15.0	0.5	0.2423	1.0654	40.0	0.5	0.2358	0.9463
15.0	1.0	0.2401	0.9692	40.0	1.0	0.2491	0.9696
15.0	5.0	0.2675	1.0161	40.0	5.0	0.2639	1.0056
15.0	10.0	0.2730	1.0305	40.0	10.0	0.2660	1.0114
15.0	15.0	0.2749	1.0359	40.0	15.0	0.2668	1.0134
15.0	20.0	0.2759	1.0386	40.0	20.0	0.2671	1.0144

The values of θ are measured in degrees. Table 1 shows that when k is high ($k \geq 5$), the LCU distribution is positively skewed for all values of λ but could be negatively skewed if $k \leq 5$). The skewness and the kurtosis decrease when k increases at a fixed λ (for all $\lambda \geq 5$). But, the skewness and the kurtosis increase when λ increases for a fixed k (for all $k \geq 5$). Note that the skewness and kurtosis of LCU distribution do not change for all values of α and θ and shown in theorem 2. We used the quantile function in equation (20) for different values of k and λ and we compute the values of S and K using equations (22) and (23) respectively. The graphs for the skewness and kurtosis for increasing values of k and for different fixed values of λ are presented in Figure 1.

Fig. 1a and Fig. 1b of Figure 1 are the Galton's skewness for LCU distribution for values of $k \leq 1$ and $k \geq 5$ respectively. Fig. 1a shows that if the values of k are less than one, the skewness can be positive or negative, also shown in Table 1; Fig. 1b shows that for increasing k for $k \geq 5$, the skewness is an increasing positive function as also shown is Table 1.

Fig. 1c and Fig. 1d of Figure 1 are the Moors' kurtosis for LCU distribution for values of $k \leq 1$ and $k \geq 5$ respectively. Fig. 1c shows that if the values of k are less than one, the kurtosis can be positive or negative, also shown in Table 1; Fig. 1d shows that for increasing k for $k \geq 5$, the kurtosis is an increasing positive function as also shown is Table 1.

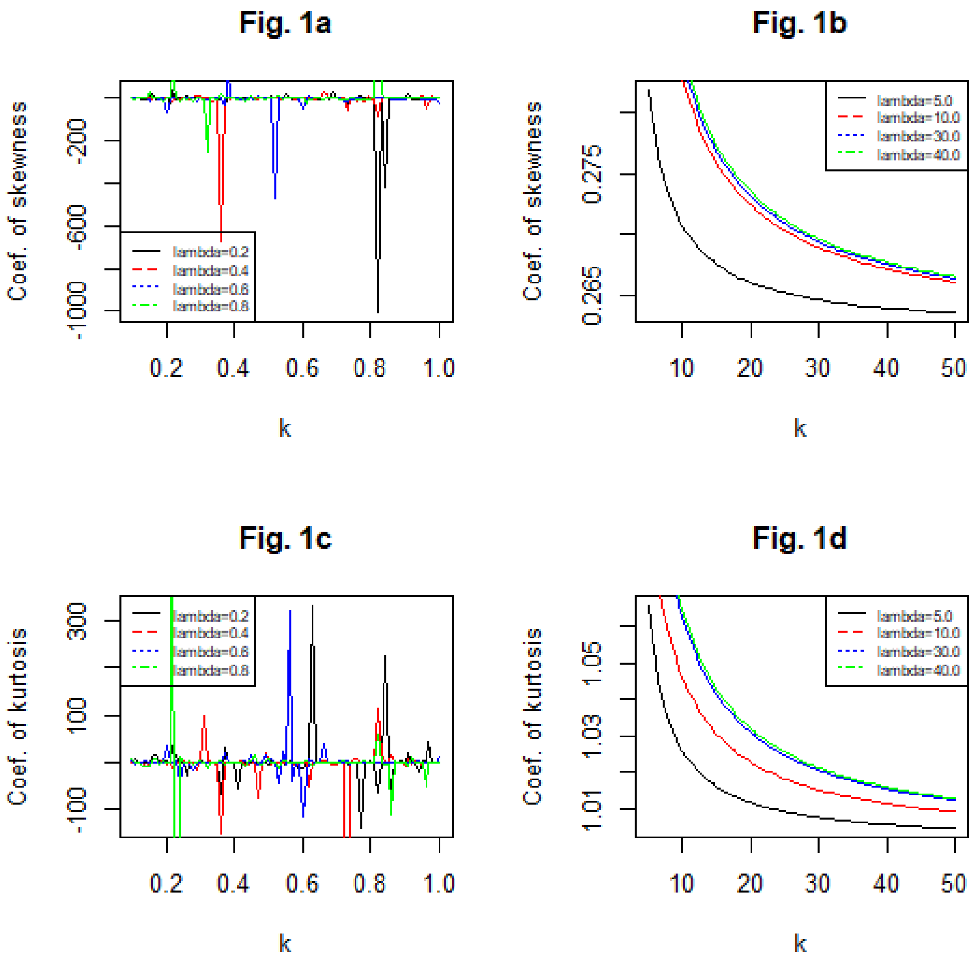

Figure 1: Galton's skewness (S) and Moors' kurtosis (K) for the LCU distribution

2.3 *Linear representation of LCU*

Recall the PDF in equation (12) given as

$$f_X(x) = \frac{\lambda k}{\alpha \pi}\left[\left(\frac{x-\theta}{\alpha}\right)^2 + 1\right]^{-1}\left\{1+\lambda\left[\frac{1}{2}+\frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]\right\}^{-k-1}, \; \alpha, k, \lambda > 0, \theta \geqslant 0, x \in \Re$$

Using binomial and Maclaurin expansion from Gradshteyn, I.S, Ryzhik, I.M(2007), we have the following series

$$\left[\left(\frac{x-\theta}{\alpha}\right)^2 + 1\right]^{-1} = \sum_{i=0}^{\infty}(-1)^i\binom{1}{i}\left(\frac{x-\theta}{\alpha}\right)^{2i} \tag{30}$$

$$\left\{1+\lambda\left[\frac{1}{2}+\frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]\right\}^{-(k+1)} = \sum_{j=0}^{\infty}(-1)^j\lambda^j\binom{k+1}{j}\left[\frac{1}{2}+\frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]^j \tag{31}$$

$$\left[\frac{1}{2}+\frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]^j = \sum_{n=0}^{\infty}(1/2)^{j-1}\binom{j}{n}\left[\frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]^n \tag{32}$$

$$tan^{-1}\left(\frac{x-\theta}{\alpha}\right) = \sum_{m=0}^{\infty}(-1)^m \frac{\left(\frac{x-\theta}{\alpha}\right)^{2m+1}}{2m+1}; \left(\frac{x-\theta}{\alpha}\right) \leq\leq 1 \tag{33}$$

Insert equations (30), (31), (32) and (33) into the pdf in (12) to obtain

$$f_X(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} w_{i,j,l,m,n}\xi_{\alpha,\lambda,\theta,k}x_i^l \tag{34}$$

where

$$w_{i,j,l,m,n} = \frac{\pi^{-(n+1)}(2m+1)^{-n}(-1)^{3i+3mn+j+n}2^{n-j}\Gamma(2)\Gamma(j+1)\Gamma(k+2)\Gamma(2i+2mn+n+1)}{\Gamma(2-i)\Gamma(i+1)\Gamma(j-n+1)\Gamma(n+1)\Gamma(k-j+2)\Gamma(j+1)\Gamma(2i+2mn+n-l+1)\Gamma(l+1)} \tag{35}$$

and

$$\xi_{\alpha,\lambda,\theta,k} = k\frac{\lambda^{j+1}\theta^{2i+2mn+n-l}}{\alpha^{2i+2mn+n+1}} \tag{36}$$

2.4 *Order statistics*

By definition, the rth order statistics random variable X is given by

$$f_{r:n} = \frac{n!}{(r-1)!(n-r)!}f_X(x)[F_X(x)]^{(r-1)}[1-F_X(x)]^{(n-r)} \tag{37}$$

Then the $f_{r:n}(x_r)$ of a LCU distribution is derived by inserting (11) into (30) and by using expansion in (30) - (33), given by

$$f_{r:n} = \frac{kn!}{\pi(r-1)!(n-r)!}\sum_{p,q=0}^{\infty}\frac{(-1)^p\binom{n-r}{p}\binom{p+r-1}{q}}{\left[\left(\frac{x-\theta}{\alpha}\right)^2+1\right]}\left\{1+\lambda\left[\frac{1}{2}+\frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]\right\}^{-(kq+k+1)} \tag{38}$$

and let

$$f_{\alpha,\lambda,\theta,(kq+k+1)}(x_r) = \left[\left(\frac{x-\theta}{\alpha}\right)^2+1\right]^{-1}\left\{1+\lambda\left[\frac{1}{2}+\frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]\right\}^{-(kq+k+1)} \tag{39}$$

with parameter α,λ ,θ and $(kq+k+1)$ we have

$$f_{r:n} = \frac{kn!}{\pi(r-1)!(n-r)!}\sum_{p,q=0}^{\infty}(-1)^p\binom{n-r}{p}\binom{p+r-1}{q}f_{\alpha,\lambda,\theta,(kq+k+1)}(x_r) \tag{40}$$

2.5 *Shannon's entropy*

The entropy of a random variable is a measure of variation of uncertainty. Let X be a random variable with probability density function f(x) where support is a set X, then the differential or Shannon's entropy of the variable X is given by

$$\eta_x = -E\{log\left(f_{LCD}(x)\right)\}$$

Using Aljarrah et al. (2014) and Alzaatreh et al. (2016) results,

$$\eta_x = \eta_T + E\{log\left(f_y(T)\right)\} - E\{log\left(f_c\left(Q_c\left(F_y(T)\right)\right)\right)\} \tag{41}$$

where η_T is the entropy of Lomax distribution

$$E\{log\left(f_c\left(Q_c\left(F_y(T)\right)\right)\right)\} = -log\left(\pi\theta\right) + 2log\left(sin\left(\pi F_y(T)\right)\right) \tag{42}$$

and

$$log\left(sin\left(\pi x\right)\right) = log\left(\pi x\right) + 2\sum_{j=1}^{n} V_j x^{2j}, \tag{43}$$

$$\eta_T = -log(k/\lambda) + \left(\frac{k+1}{k}\right) \tag{44}$$

Substitute (42) in (41) and further substituting (41) and (42) in (40), we have the Shannon entropy of LCD

$$\eta_x = -log\left(\pi\right) + log\left(\theta\right) - log(k/\lambda) + \left(\frac{k+1}{k}\right) + E\{log\left(f_y(T)\right)\} - 2E\left\{log\left(x\right)\right\} - 2\sum_{j=1}^{n} V_j x^{2j} \tag{45}$$

where $V_j = \dfrac{(-1)(2\pi)^2 j B_{2j}}{2j(2j)!}$ and B_j is the Bernoulli number Alzaatreh et al. (2016).

2.6 *Maximum likelihood estimation*

By definition, the likelihood of a random variable X with PDF $f_X(x)$ and a parameter space ξ is given by

$$L(x, \xi) = \Pi_{i=1}^{n} f(x, \xi) \tag{46}$$

Put the PDF in (11) into equation (38) and take the log to arrive at the loglikelihood function of LCU distribution given by

$$l = nln(\lambda) + nln(k) - nln(\pi\alpha) - \sum_{i=1}^{n} ln\left[\left(\frac{x-\theta}{\alpha}\right)^2 + 1\right] - (k+1)\sum_{i=1}^{n} ln\left\{1 + \lambda\left[\frac{1}{2} + \frac{1}{\pi}tan^{-1}\left(\frac{x-\theta}{\alpha}\right)\right]\right\} \tag{47}$$

where $l = logL(x, \xi)$. Differentiate equation (39) partially with respect to each of the parameters and equate the results to zero.

$$\frac{\delta l}{\delta\alpha} = \frac{n}{\alpha} - \sum_{i=0}^{n} -\frac{2\left(\frac{x_i-\theta}{\alpha}\right)^2}{\alpha^3\left[\left(\frac{x_i-\theta}{\alpha}\right)^2 + 1\right]} - (k+1)\sum_{i=0}^{n}\left(-\frac{\lambda\left(x_i-\theta\right)}{\pi\alpha^2\left\{1 + \lambda\left[\frac{1}{2} + \frac{1}{\pi}tan^{-1}\left(\frac{x_i-\theta}{\alpha}\right)\right]\right\}}\right) \tag{48}$$

$$\frac{\delta l}{\delta\lambda} = \frac{n}{\lambda} - (k+1)\sum_{i=0}^{n}\left(\frac{\left[\frac{1}{2} + \frac{1}{\pi}tan^{-1}\left(\frac{x_i-\theta}{\alpha}\right)\right]}{\left\{1 + \lambda\left[\frac{1}{2} + \frac{1}{\pi}tan^{-1}\left(\frac{x_i-\theta}{\alpha}\right)\right]\right\}}\right) \tag{49}$$

$$\frac{\delta l}{\delta k} = \frac{n}{k} - \sum_{i=1}^{n} ln\left\{1 + \lambda\left[\frac{1}{2} + \frac{1}{\pi}tan^{-1}\left(\frac{x_i-\theta}{\alpha}\right)\right]\right\} \tag{50}$$

$$\frac{\delta l}{\delta\theta}=-\sum_{i=0}^{n}-\frac{2\left(\frac{x_i-\theta}{\alpha}\right)}{\alpha^2\left[\left(\frac{x_i-\theta}{\alpha}\right)^2+1\right]}-(k+1)\sum_{i=0}^{n}\left(-\frac{\lambda}{\pi\alpha\left\{1+\lambda\left[\frac{1}{2}+\frac{1}{\pi}tan^{-1}\left(\frac{x_i-\theta}{\alpha}\right)\right]\right\}}\right) \tag{51}$$

Solving them simultaneously, $\hat{\alpha}$, $\hat{\lambda}$, $\hat{k}$ and $\hat{\theta}$, which are the maximum likelihood estimates (MLEs) for the parameters α, λ, k and θ are respectively obtained. The computations are done using the maxLik package in R 3.3.2 software.

3. Results and discussion

3.1 *Simulation*

In this section, the relation between lomax-Cauchy and lomax random variables in equation (17) was used to simulate data for LCU distribution. The random variates of lomax was generated and then inserted into the relation in (17) to generate the LCU random variates. The maximum likelihood estimates are computed for each simulated sample. Results on the biases (estimate – actual) and the standard deviations of the estimates are used to investigate the consistency and performance of the method of maximum likelihood estimation. The simulation was carried out for various values of the parameters of the LCU distribution and the results are similar. We reported the results for the parameter values λ = 0.5, 1.5, 2.0, 2.5; θ = 30, 45, 60, 65; k= 0.5 and α = 1. Four different sample sizes of n = 100, 500 and 1000 are used in the simulation. For each sample size and each parameter combination, the maximum likelihood estimates of the parameters λ, θ, k, and α are computed and the process is repeated 500 times. The average bias and the standard deviation of the maximum likelihood estimates are computed. The results are reported in Table 2.

Table 2: Maximum likelihood estimates, bias and standard error of difference combination of values of parameters (λ, θ, k) when parameter $\alpha = 1$.

	n=20				n=100			n=500			n=1000		
Parameter	Actual	Estimate	Bias	Std. Error	Estimate	Bias	Std. Error	Estimate	Bias	Std. Error	Estimate	Bias	Std. Error
λ	0.5	0.401	0.099	0.201	0.218	0.282	0.099	0.329	0.171	0.094	0.489	0.011	0.056
θ	30.	29.917	0.083	0.157	29.867	0.133	0.032	29.862	0.138	0.025	29.993	0.007	0.068
K	0.5	0.292	0.208	0.436	0.166	0.334	0.088	0.143	0.357	0.062	0.491	0.009	0.197
λ	1.5	1.336	0.164	0.448	1.736	0.236	0.368	1.687	0.187	0.291	1.471	0.029	0.156
θ	45	45.121	0.121	4.528	39.014	0.986	NA	46.695	1.695	1.407	45.053	0.053	0.645
K	0.5	0.594	0.094	2.230	0.008	0.492	NA	1.161	0.661	0.502	0.535	0.035	0.301
λ	2.0	1.781	0.219	0.591	1.640	0.360	0.442	2.214	0.214	0.330	1.962	0.038	0.208
θ	60	60.070	0.070	1.966	61.650	1.650	3.365	61.616	1.616	1.151	60.046	0.046	0.568
K	0.5	0.555	0.055	0.607	1.052	0.552	0.943	0.892	0.392	0.270	0.524	0.024	0.166
λ	2.5	2.225	0.275	0.757	2.047	0.453	0.525	2.756	0.256	0.417	2.454	0.046	0.258
θ	65	65.028	0.028	2.616	66.673	1.673	2.128	66.665	1.665	1.238	65.044	0.044	0.524
K	0.5	0.540	0.040	0.572	0.909	0.409	0.420	0.787	0.287	0.205	0.519	0.019	0.112

Table 2 shows that the maximum likelihood estimation method performs well in estimating the LCU distribution parameters. The estimates of the parameters and their standard errors are reasonable and consistent. They show that the standard errors of the estimates decrease as the sample size increases. However, the biases, which are somewhat small, do not show a clear decreasing trend as sample size increases. For fixed λ, θ and k the bias and the standard error for the estimate of λ increases as n increases. Similarly, for fixed λ, θ and k the bias and the standard error for the estimate of k increases as n increases. The results from this research, suggest that the maximum likelihood method can be used to estimate the parameters of the LCU distribution.

3.2 *Application*

The Wheaton river data Alshawarbehet et al. (2013) used a dataset on the exceedances of flood peaks (m^3/s) of the Wheaton River in Yukon Territory, Canada. Which was previously applied by Akinsete et al (2008) to fit generalized Pareto distribution.

Table 3. Exceedances of the Wheaton River data

1.7	2.2	14.4	1.1	0.4	21	5.3	0.7	1.9	13	12	9.3
1.4	19	8.5	26	12	14	22.1	1.1	2.5	14	2	38
0.6	2.2	39	0.3	15	11	7.3	23	1.7	0.1	1	0.6
9	1.7	7	20	0.4	2.8	14.1	9.9	10	11	30	3.6
5.6	31	13.3	4.2	26	3.4	11.9	22	28	36	3	64
1.5	2.5	27.4	1	27	20	16.8	5.3	9.7	28	3	27

We therefore applied Cauchy distribution, Beta Pareto, beta-Cauchy distributions and the new Lomax Cauchy distribution to fit the dataset. The result is presented below in Table 4.

Table 3: Parameter estimates (standard errors in parentheses) for the Wheaton River data

Model	Cauchy*	Beta-Pareto*	Beta-Cauchy*	Lomax Cauchy
Parameter Estimates	$\hat{\theta} = 7.0718$ (1.4300)	$\hat{\alpha} = 7.6954$	$\hat{\alpha} = 387.6492$ (437.7976)	$\hat{\alpha} = 23.342$ (1.482)
	$\hat{\lambda} = 6.4622$ (0.9825)	$\hat{\beta} = 85.75$	$\hat{\beta} = 1.4589$ (0.5127)	$k = 11.024$ (0.431)
		$\hat{\lambda} = 0.0208$	$\hat{\theta} = -2.0467$ (1.2725)	$\hat{\theta} = 0.187$ (0.987)
		$\hat{\theta} = 0.1$	$\hat{\lambda} = 0.0787$ (0.0963)	$\lambda = 0.213$ (0.193)
K-S	0.2240	0.1486	0.1219	0.0873
K-S p-value	0.0015	0.0831	0.2347	0.4782
Log likelihood	-288.4709	-272.1280	-260.4802	-249.3422
AIC	580.9	552.256	528.96	506.6844

$^*Alshawarbeh\ et\ al.$ (2013).

We fit the LCU distribution to the Wheaton river data and compared with result from Alshawarbehet al (2013). Maximum likelihood estimation method as discussed in Section 2.6 is adopted to estimate the unknown parameters α, β, k, θ . The computations were done using the maxLik function in R. Table 4 lists the MLEs (and the corresponding standard errors in parentheses) of the model parameters, the log-likelihood value, the AIC (Akaike Information Criterion), the Kolmogorov- Smirnov (K-S) test statistic, and the p-value for the K-S statistic for the fitted distributions. These results show that the LCUD models have the lowest log likelihood and AIC, and therefore it could be selected as the best model.

4. Conclusion

In this paper, we propose a new lifetime model called the Lomax-Cauchy(LCD)distribution. We study some of its structural properties including an expansion for the density function and quantile function and order statistic. The maximum likelihood method is employed for estimating the model parameters and the observed information matrix is determined. The histogram using LCU distribution variates generated using the quantile function, empirical density plot, empirical CDF, empirical survival, empirical hazard and empirical cumulative hazard plots are depicted in Figures 2, 3, 4, 5, 6 and 7 respectively to show the shape of the distribution at different parameter values (see Appendix 1). Order statistic and Shannon entropy of LCU distribution were studied. We fit the new model to one real data sets on exceedances of flood peaks of Wheaton River to demonstrate its usefulness in practice. We conclude that the LCD model provides consistently better fits than other competing models for the exceedances of flood peaks of Wheaton River data. We hope that the proposed model will attract wider applications in areas such as environmental hazard, engineering, survival and lifetime data, hydrology, economics, among others.

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Appendix

Appendix 1

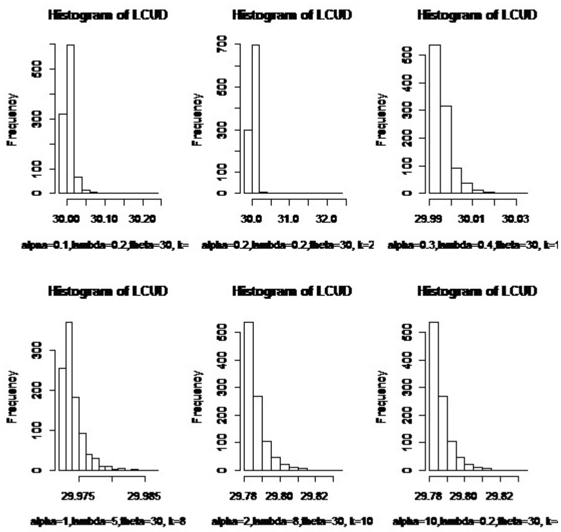

Figure 2: Histogram of LCU distribution

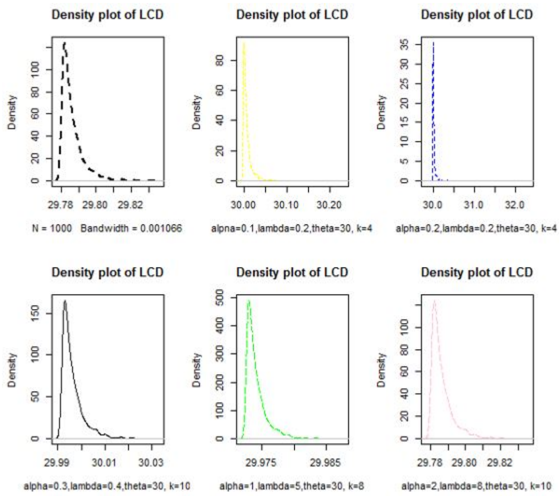

Figure 3: Density plot of LCU distribution

Appendix 2 R Code

LomaxCauchy Distribution
set.seed(12345)
n = 20
alpha = 1
lambda = 0.5
theta = 30
k = 0.5
p = sort(runif(n))
y = rlomax(p, lambda, k)
x = theta + alpha*tan('pi*(y-0.5))
lcu=function(param,x) alpha=param[1]
lambda=param[2]
theta=param[3] k=param[4]
$L = n * log(lambda) + n * log(k) - n * log(pi) - n * log(alpha) - sum'(log(((x - theta)/alpha)^2 + 1)) - (k +$

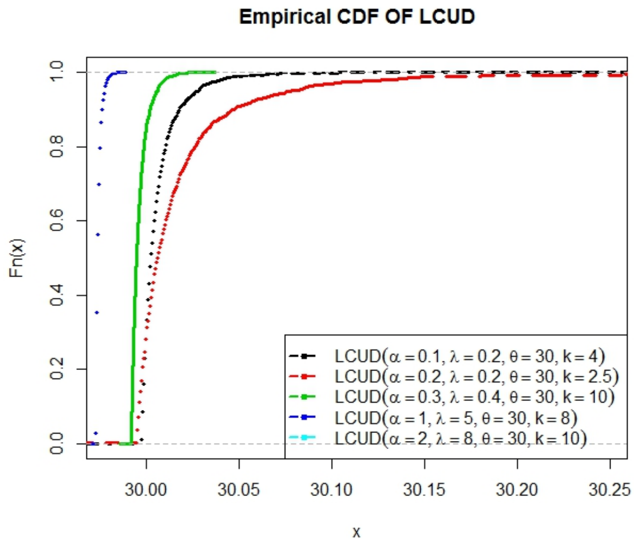

Figure 4: Empirical CDF of LCU

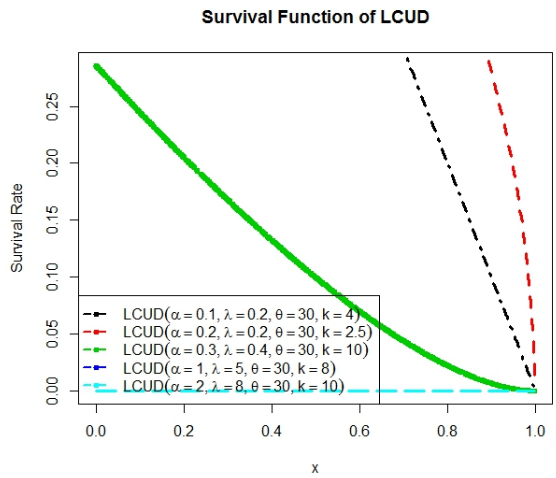

Figure 5: Empirical survival of LCU distribution

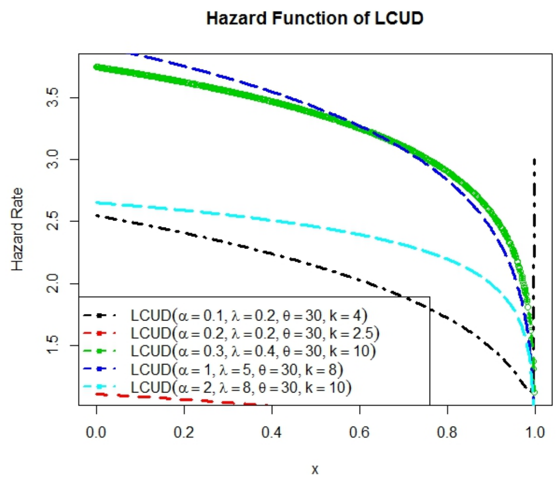

Figure 6: Empirical hazard of LCU distribution

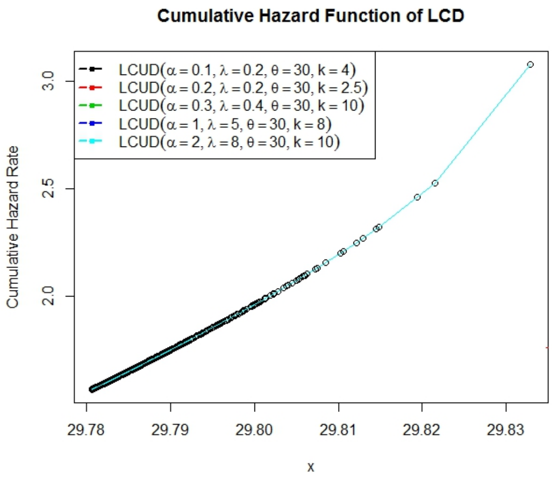

Figure 7: Empirical cumulative hazard of LCU distribution

$1) * sum(log(1 + lambda * (0.5 + (1/pi') * (180/3.14159265) * atan((x - theta)/alpha))))$
return((L))
Est= maxLik(lcu, x=x,start=c(1,1,1,1))
summary(Est)

$rm(list = ls())$
$set.seed(12)$
$p = sort(runif(1000, 0, 1))$
$alpha = 0.1$
$lambda = 0.2k = 4$
$tita = 30$
$y1 = alpha * tan((pi * (((1 - p)^(- 1/k) - 1)/lambda - 1/2)) * 3.1'4159265/180) + tita$
$summary(y1)$
$S1 = (1 + lambda * (1/2 + 1/pi * (180/3.14159265) * atan((y1 - tita)/alpha)))^(- k)$
$h1 = lambda*k/(alpha*pi)*((((y1-ti'ta)/alpha)^(2)+1)^(-1)*(1+lambda*(1/2+1/pi*(180/3.14159265)*$
$atan((y1 - tita)/alpha))))^(- 1)$
$H1 = log(1 + lambda * (1/2 + 1/pi * (180/'3.14159265) * atan((y1 - tita)/alpha)))$
$alpha = 0.2$
$lambda = 0.3$
$k = 2.5$
$tita = 30$
$q = (1 - p)^(- 1/k)$
$y2 = alpha * tan((pi * (((1 - p)^(- 1/k') - 1)/lambda - 1/2)) * 3.14159265/180) + tita$
$summary(y2)$
$S2 = (1 + lambda * (1/2 + 1/pi * (180/3.14159265) * atan((y1 - tita)/alpha)))^(- k)$
$h2 = lambda*k/(alpha*pi)*((((y1'-tita)/alpha)^(2)+1)^(-1)*(1+lambda*(1/2+1/pi*(180/3.14159265)*$
$atan((y1 - tita)/alpha))))^(- 1)$
$H2 = log(1 + lambda * (1/2 + 1/pi * (1'80/3.14159265) * atan((y1 - tita)/alpha)))$
$alpha = 0.3$
$lambda = 0.4$
$k = 10$
$tita = 30$
$q = (1 - p)^(- 1/k)$
$y3 = alph * tan((pi * (((1 - p)^(- 1/k) - 1)/lambda - 1/2)) * 3.14159265/180) + tita$
$summary(y3)$
$S3 = (1 + lambda * (1/2 + 1/pi * (180/3.14159265) * atan((y1 - tita)/alpha)))^(- k)$
$h3 = lambda*k/(alpha*pi)*(((('y1-tita)/alpha)^(2)+1)^(-1)*(1+lambda*(1/2+1/pi*(180/3.14159265)*$
$atan((y1 - tita)/alpha))))^(- 1)$
$H3 = log(1 + lambda * (1/2 + 1/pi *' (180/3.14159265) * atan((y1 - tita)/alpha)))$
$alpha = 1$
$lambda = 5$
$k = 8$
$tita = 30$
$q = (1 - p)^(- 1/k)$

$y4 = alpha * tan((pi * (((1 - p)^{(} - 1/k) - 1)/lambda - 1/2)) * 3.14159265/180) + tita$
$summary(y4)$
$S4 = (1 + lambda * (1/2 + 1/pi * (180/3.14159265) * atan((y1 - tita)/alpha)))^{(} - k)$
$h4 = lambda * k/(alpha * pi) * ('(((y1 - tita)/alpha)^{(}2) + 1)^{(} - 1) * (1 + lambda * (1/2 + 1/pi * (180/3.14159265) * atan((y1 - tita)/alpha))))^{(} - 1)$
$H4 = log(1 + lambda * (1/2 + 1/'pi * (180/3.14159265) * atan((y1 - tita)/alpha)))$
$alpha = 2$
$lambda = 8$
$k = 10$
$tita = 30$
$q = (1 - p)^{(} - 1/k)$
$y5 = lambda * tan((pi * (((1 - p)^{(} - 1/k) - 1)/lambda - 1/2)) * 3.14159265/180) + tita$
$summary(y5)$
$S5 = (1 + lambda * (1/2 + 1/pi * (180/3.14159265) * atan((y1 - tita)/alpha)))^{(} - k)$
$h5 = lambda * k/(alpha * pi) * ((((y1 - tita)/alpha)^{(}2) + 1)^{(} - 1) * (1 + lambda * (1/2 + 1/pi * (180/3.14159265) * atan((y1 - tita)/alpha))))^{(} - 1)$
$H5 = log(1 + lambda * (1/2 + 1/pi * (180/3.14159265) * atan((y1 - tita)/alpha)))$
$par(mfrow = c(2, 3))$
$hist(y1, xlab = "alpna = 0.1, , lambda = 0.2, theta = 30, k = 4", main = "HistogramofLCUD")$
$hist(y2, xlab = "alpha = 0.2, , lambda = 0.2, theta = 30, k = 2.5", main = "HistogramofLCUD")$
$hist(y3, xlab = "alpha = 0.3, , lambda = 0.4, theta = 30, k = 10", main = "HistogramofLCUD")$
$hist(y4, xlab = "alpha = 1, , lambda = 5, theta = 30, k = 8", main = "HistogramofLCUD")$
$hist(y5, xlab = "alpha = 2, , lambda = 8, theta = 30, k = 10", main = "HistogramofLCUD")$
$hist(y5, xlab = "alpha = 10, , lambda = 0.2, theta = 30, k = 4", main = "HistogramofLCUD")$