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The use of the weighted least squares method when the error variance is heteroscedastic

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Abstract. Homoscedastic property of a variance of the errors in a linear regression is among the assumptions of the Ordinary Least Square (OLS) method. When this assumption of homoscedasticity is violated, it causes the regression coefficients to be biased and inconsistent. Various methods have been used in the literature to detect the presence of heteroscedasticity. This study compares two of the existing methods of detecting the presence of heteroscedasticity. The two methods are; Goldfeld-Quandt (GQ) and Breusch-Pagan-Gofrey (BP) test. Results show that the GQ test is better than the BPG test in terms of their P-values. In the presence of heteroscedasticity, the study adopts the method of Weighted Least Squares (WLS) to circumvent the problems of associated with heteroscedasticity.

Keywords: homoscedasticity, heteroscedasticity, variance, Goldfeld-Quandt test and Breusch-Pagan test.

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1. Introduction

Regression Analysis are widely used in different aspects of life (Usman, 2015). Classical linear regression model assumes that the error term e_i in the regression model is homoscedastic (i.e. equal variance) across the observations. However, if assumption of homoscedasticity, or equal variances, is not satisfied, we then have the problem of heteroscedasticity (unequal variance). Among the consequences of heteroscedasticity are that, the OLS estimates are unbiased and consistent, they may no longer have efficient or minimum variance, and they seized to be best linear unbiased estimators (BLUE). However, in the presence of heteroscedasticity the BLUE estimators are provided by the method of weighted least squares. Due to the presence of heteroscedasticity, the t and F test under standard assumptions may not be reliable.

Various methods were proposed in the literature to detect the presence of heteroscedasticity. Among the formals tests are: white test (White, 1980), Breusch-Pagan (Breusch and Pagan, 1979), Park test (Park, 1969), Glejser test (Glejser, 1969), $Spearman's$ rank correlation test, Goldfeld-Quandt Test. (Goldfeld and Quandt, 1980) and Koenker-Bassett (KB) test.

Researchers have continued to excellently investigate and compared different tests of heteroscedasticity. According to (Long and Ervin, 1998) white test has low power for small sample:- This was shown by exploring the small sample properties of four versions of HCCM (Heteroscedasticity-Consistent Covariance Matrix) in a linear regression model are HC0, HC1, HC2, and HC3. Among them the HC3 was the least powerful, followed by HC2 and HC1. However, the differences were greatly reduced after adjustment of power for size distortion. A comparison between $Szroeter's$ asymptotic test and Goldfeld-Quandt (GQ) test. (Goldfeld and Quandt, 1980), Breusch-Pagan test (Breusch and Pagan, 1979) and BAMSET (Ramsey, 1969) was conducted by Griffiths and Surekha (1985). Goldfeld-Quandt test being the most popular and performed satisfactorily. Breusch-Pagan (BPG) test is also popular and powerful. The BAMSET is less sensitive. Griffiths and Surekha:

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(1985) examined the tests for both large and small samples among the four tests. Feng; (1981) studied Bayesian and frequentist approach for heteroscedasticity. Among the frequentist approach, BPG test, white test and Koenker-Basset test (Koenker and Bassett, 1982) were compared. The study found that the white test has the weakness of identifying the variance that creates the problem of heteroscedasticity. Koenker test fails to detect the problem when sample sizes are small due to a large degree of freedom. Yüce; (2008) introduces a diagnostic that do not rely on the assumption of heteroscedasticity and does require estimation of an auxiliary regression model. The test is asymptotic on ordinary least square. Midi; (2008) proposed a modified GQ test for the detection of heteroscedasticity when there are outliers. Using two different sets of data, the conventional test such as white test, BPG test and GQ test failed to detect the heteroscedasticity in the presence of outliers unlike the modified GQ test.

This paper uses a data set on forestry, fishing, livestock and crop production. The data were sourced from National Bureau of Statistics (NBS); it covers the period from 1981-2010.

The paper compares two conventional tests that are widely used by econometricians namely: Breusch-Pagan (Breusch and Pagan, 1979) and Goldfeld-Quandt test. (Goldfeld and Quandt, 1980).

1.1 *Screening of heteroscedasticity graphically*

Graphical method is an informal method of detecting the presence of heteroscedasticity. The residuals are usually plotted against the fitted variables or any of the explanatory variables. The distribution/spreads of the residuals to a non-systematic patterns indicates the presence of heteroscedasticity (Chatterjee, 2006).

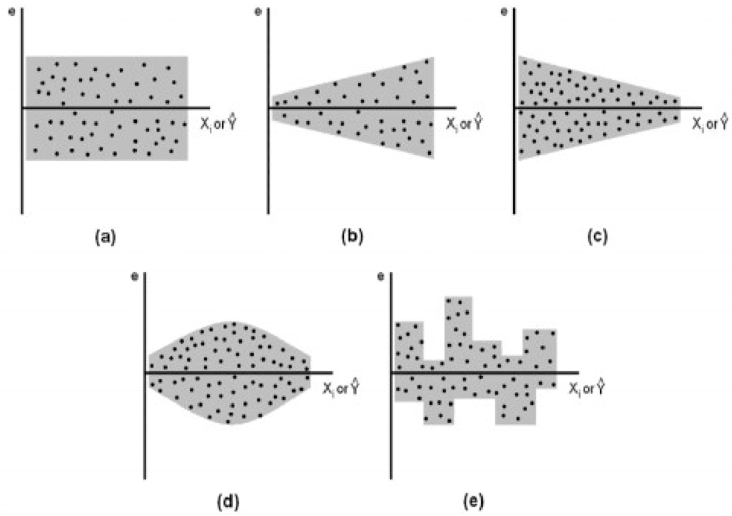

Figure 1.: Error variance distribution

Figure 1 illustrates some graphs of the residuals against independent variable or fitted values. Fig. 1 (a) shows a homoscedastic residual structure. Fig. 1(b) and Fig. 1(c) shows outward-opening and inward-opening funnel patterns respectively, which are the indicators of heteroscedasticity structure Fig. 1(d) depicts the elliptic shape heteroscedasticity structure. Fig. 1(e) shows the irregular type of heteroscedasticity. In this case, variances of some disturbances are different from others.

2. Materials and methods

2.1 *Hypothesis*

H_0: *The error variance is homoscedastic*
versus
H_1: *The error variance is not homoscedastic*

2.2 *Formal tests*

2.2.1 Breusch Pagan test

To illustrate this test, consider the k-variable linear regression model

$$Y_i = \beta_1 + \beta_2 X_{2i} + ... + \beta_k X_{ki} + \mu_i \tag{1}$$

Assume that the error variance σ_I^2 described as

$$\sigma_I^2 = f(\propto_1 + \propto_2 Z_{2i} + ...+ \propto_m Z_{mi}) \tag{2}$$

That is, σ_I^2 is some function of the non-stochastic variables $Z^{'}s$ (it is assumed that predictor variable is stochastic in nature and regressor variables are non-stochastic in nature); some or all of the $X^{'}s$ can serve as $Z^{'}s$. Specifically, assume that

$$\sigma_I^2 = f(\propto_1 + \propto_2 Z_{2i} + ...+ \propto_m Z_{mi}) \tag{3}$$

that is, σ_I^2 is a linear function of the $Z^{'}s$. If $\propto_2 = \propto_3 =$ .... $= \propto_m = 0, \sigma_I^2 = \propto_1$, then the variance is a constant. Therefore, to test whether σ_I^2 is homoscedastic, one can test the hypothesis that $\alpha_2 = \alpha_3 = ... = \alpha_m = 0$. This is the basic idea behind the Breusch - Pagan test. The actual test procedure is as follows (Gujarati, 2004).

Step 1. Estimate (1) by OLS and obtain the residuals $\hat{\mu}_1, \hat{\mu}_2, ..., \hat{\mu}_n$.
Step 2. Obtain $\hat{\sigma}^2 = \sum \hat{\mu}_i^2/n$. But this ML estimator of σ^2
Step 3. Construct variables p_i defined as

$$p_i = \hat{\mu}_i^2/\sigma^2$$

Step 4. Regress p_i on the $Z^{'}s$ as

$$p_i = \propto_1 + \propto_2 Z_{2i} + ...+ \propto_m Z_{mi} + V_I \tag{4}$$

where V_I is the residual term of the regression.
Step 5. Obtain the ESS (explained sum of squares) from (4) and define

$$T = SSE/2$$

Assumingμ_i are normally distributed, one can show that if there is homoscedasticity and if the sample size, n, is large, then

$$T \sim \chi^2_{m-1} \tag{5}$$

Therefore, if in an application the computed T exceeds the critical χ^2 value at the chosen level of significance, one can reject the hypothesis of homoscedasticity; otherwise one does not reject it.

2.2.2 Goldfeld-Quandt test

This popular test is applicable when one assumes that the heteroscedastic variance, σ_I^2 is positively related to one of the explanatory variables in the regression model. For simplicity, consider the usual two-variable model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + ... + \beta_k X_{ki} + \mu_i \tag{6}$$

Suppose σ_I^2 is positively related to X_i as

$$\sigma_I^2 = \sigma^2 X_i^2 \tag{7}$$

where σ^2 is a constant.

Assumption (7) postulates that σ_I^2 is proportional to the square of the X variable. If (7) is appropriate, it would mean σ_I^2 would be larger, the larger the values of X_i. If that turns out to be the case, heteroscedasticity is most likely to be present in the model. To test this explicitly, Goldfeld and Quandt suggest the following steps (Gujarati, 2004):

Step 1. Order or rank the observations according to the values of X_i, beginning with the lowest X value.

Step 2. Omit c central observations, where c is specified a priori, and divide the remaining (n - c) observations into two groups each of (n - c)/2 observations.

Step 3. Fit separate OLS regressions to the first (n - c)/2 observations and the last (n - c)/2 observations, and obtain the respective residual sums of squares RSS1and RSS2, RSS1 representing the RSS from the regression corresponding to the smaller X_i values (the small variance group) and RSS2 that from the larger X_i values (the large variance group). These RSS each have (Gujarati, 2004)

$$\frac{n-c}{2} - K \quad or \quad \left(\frac{n-c-2k}{2}\right) df$$

where k is the number of parameters to be estimated, including the intercept.

Step 4. Compute the ratio

$$\lambda = \frac{RSS_1/df}{RSS_2/df} \tag{8}$$

If μ_i are assumed to be normally distributed (which we usually do), and if the assumption of homoscedasticity is valid, then it can be shown that λ of (8) follows the F distribution with numerator and denominator df each of (n-c-2k)/2.
If in an application the computed $\lambda(= F)$ is greater than the critical F at the chosen level of significance, we can reject the hypothesis of homoscedasticity, that is, we can say that heteroscedasticity is very likely.

2.3 *Remedial measure*

2.3.1 Weighted least square

Heteroscedastic errors can be corrected by either transforming the predictor variable (Weisberg, 1980) or by either transforming both sides. Also, the WLS method is presented here as a way of dealing with heteroscedastic errors. Recall the linear regression equation

$$E(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_k X_k \tag{9}$$

We have estimated the parameters $\beta_0, \beta_1, ..., \beta_k$ by minimizing the sum of squared residuals

$$SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$
$$= \sum_{i=1}^{n}(y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_{i,1} + \hat{\beta}_2 X_{i,2} + ... + \hat{\beta}_k X_{i,k}))^2$$

Sometimes we want to give some observations more weight than others. We achieve this by minimizing a weighted sum of squares:

$$WSSE = \sum_{i=1}^{n} w_i(y_i - \hat{y}_i)^2$$

$$WSSE = \sum_{i=1}^{n} w_i(y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_{i,1} + \hat{\beta}_2 X_{i,2} + ... + \hat{\beta}_k X_{i,k}))^2$$

The resulting $\beta' s$ are called weighted least squares (WLS) estimates, and the WLS residuals are (Chatterjee, 2006)

$$\sqrt{w_i}(y_i - \hat{y}_i)$$

Conducting OLS on the transformed variables is equivalent to WLS method.

2.3.2 Why use weights?

Suppose that the variance is not constant:

$$Var(y_i) = \sigma_i^2$$

Then we use weights

$$w_i \propto= \frac{1}{\sigma_i^2}$$

The WLS estimates have smaller standard errors than the ordinary least squares (OLS) estimates. That is, the OLS estimates are inefficient, relative to the WLS estimates. When you specify weights, regression software calculates standard errors on the assumption that they are proportional to $1/\sigma_i^2$. However, like the OLS method, the WLS regression is also sensitive to the presence of outliers. one weakness of the WLS method is that, the weights determination is much affected by the presence of outliers. The Parameter estimation is also affected and other aspects of WLS, if care is not taken.

Kutner et al (2004) proposed an estimator that can be applied to multiple regressors to deals with unequal variance, but the estimator is opposed by an outlier (Huber, 1981), the weight function curtails the effect of an outlier.

A two-step robust weighted least square was proposed by Midi (2016). The estimator take care both heteroscedasticity and outliers at the same time with a multiple regressor variables. The estimator combines estimator proposed by Kutner and Huber.

3. Results and discussions

We now present the data (see Appendix) on our variables i.e. GDP, Crop Production, Forestry, livestock and Fishery. The graph in figure 2 is showing a heteroscedastic graph

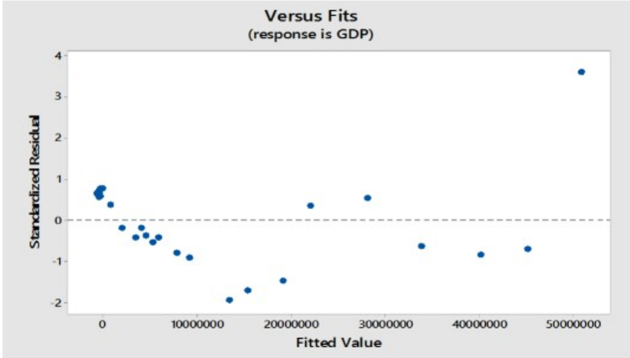

Figure 2.: Standardized residual vs. fitted value plot using MINITAB software we now proceed to conduct the respective tests on the dataset.

3.1 *Breusch-Pagan Test*

Since BPG test is sensitive to normality assumption, one needs to test its normality. Figure 3 is showing how the data was approximately distributed normally

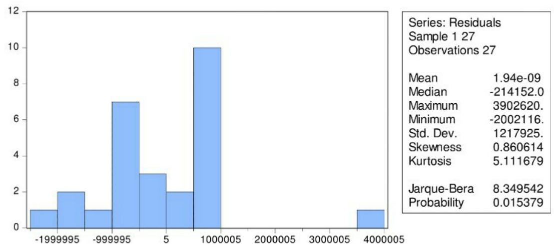

Figure 3.: Histogram for the dataset

Table 1.: Heteroskedasticity Test: Breusch-Pagan-Godfrey.

F-statistic	2.868191	Prob. F(4,22)	0.0472
$Obs * R - squared$	9.254229	Prob. Chi-Square(4)	0.0550
Scaled explained SS	12.63128	Prob. Chi-Square(4)	0.0132
R-squared	0.342749	Mean dependent var	1.43E+12
Adjusted R-squared	0.223249	S.D. dependent var	2.95E+12
S.E. of regression	2.60E+12	Akaike info criterion	60.17755
Sum squared resid	1.49E+26	Schwarz criterion	60.41752
Log likelihood	-807.3969	Hannan-Quinn criter.	60.24890
F-statistic	2.868191	Durbin-Watson stat	1.428035
Prob(F-statistic)	0.047161		

Remark: the BPG test in Table 1 above shows a p-value of 0.047 indicating a significant variable at 5 percent level of significance. thus, showing the absence of homoscedasticity. The AIC and SIC of 60.18 and 60.41 respectively, also indicate a significant value.

3.2 *Goldfeld-Quandt test*

For $n_1 = 12$, $n_2 = 12$, the three middle observations were omitted, according to GQ procedure
P- value $= 0.0030$, $\lambda = 387.39$

Remark: P-value of 0.003 is indicating the absence of homoscedasticity.

Table 2.: Comparative analysis

Tests	P - Value
Goldfeld - Quandt	0.0030
Breusch Pagan	0.0472

3.2.1 *Comparative analysis*

Remark: the results in the table 2 above shows the p-value of BPG and GQ test. GQ test has a lower p-value of 0.003 than that of BPG that is having p-value of 0.0472. GQ test having the lowest p-value considered more significant than BPG test.

3.2.2 *Remedy*

The WLS method was applied as a remedial measure for a small sample size. After running WLS method we again test the model for equal variance using BPG test

Table 3.: Weighted Statistics

R-squared	0.988163	Mean dependent var	813600.1
Adjusted R-squared	0.986011	S.D. dependent var	74428.40
S.E. of regression	80468.68	Akaike info criterion	25.59470
Sum squared resid	1.42E+11	Schwarz criterion	25.83467
Log likelihood	-340.5284	Hannan-Quinn criter.	25.66606
F-statistic	459.1362	Durbin-Watson stat	0.524243
Prob(F-statistic)	0.000000	Weighted mean dep.	274563.0

The estimate is highly significant with a p-value of 0.0000. This shows that heteroscedasticity has been dealt with.

Table 4.: Heteroskedasticity Test: Breusch-Pagan-Godfrey.

F-statistic	1.697418	Prob. F(4,22)	0.1865
$Obs * R - squared$	6.367602	Prob. Chi-Square(4)	0.1733
Scaled explained SS	3.750843	Prob. Chi-Square(4)	0.4408
R-squared	0.235837	Mean dependent var	5.28E+09
Adjusted R-squared	0.096898	S.D. dependent var	7.16E+09
S.E. of regression	6.81E+09	Akaike info criterion	48.28567
Sum squared resid	1.02E+21	Schwarz criterion	48.52564
Log likelihood	-646.8566	Hannan-Quinn criter.	48.35703
F-statistic	1.697418	Durbin-Watson stat	1.301574
Prob(F-statistic)	0.186487		

Remark: Table 4 is showing a BPG test for the corrected data with a p-value of 0.18165. The AIC and SIC of 48.29 and 48.53 respectively, also indicate a significant value. This confirm that the model is now free from heteroscedasticity.

4. Conclusions

In this paper, an empirical study was conducted base on two methods of detecting the presence of heteroscedasticity using small sample size. The study conducted BPG and GQ test on the dataset, both methods detected an unequal variance of errors. We compare the two methods base on their p-value and discover that GQ test performs better. The unequal variance of errors was removed by the WLS method. The equality of variance was tested after the applying WLS. It is therefore recommended that, when a sample size is small, GQ test is more preferable. However, comparison of GQ and BP test when sample size are large in the presence of outlier and high leverage point was reserved for future research.

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Table 1.: Appendix.

Year	Growth	GDP	Crop Production	Livestock	Forestry	Fishing
1984	4.53	170377.78	21497.55	6619.81	1379.84	867.99
1985	12.85	192273.27	25066.49	7162.61	1467.5	540.48
1986	5.29	202436.23	25972.39	7389.41	1571.76	769.07
1987	23.22	249439.08	39658.65	8373.79	1589.73	664.76
1988	28.42	320328.54	61848.89	88889.89	1859.97	1166
1989	30.86	419196.39	71883.92	11790.99	2172.69	2414.52
1990	19.2	499676.85	86926.2	14145.87	2346.08	3208.54
1991	19.29	596044.69	101645.81	15576.05	2436.64	3577.15
1992	52.64	909803.31	153379.79	23027.48	2991.28	4717.10
1993	38.39	1259070.46	249195.93	36575.99	3966.44	5586.23
1994	40.01	1762812.82	377308.29	54304.41	5982.25	7677.95
1995	64.24	2895201.36	670177.59	97202.29	8253.69	14508.06
1996	30.53	3779133.07	906894.16	130407.84	10368.68	22844.05
1997	8.80	4111640.63	1026291.49	145029.53	12554.40	27586.54
1998	11.61	4588989.84	1133389.05	158314.25	15881.38	33456.21
1999	15.65	5307361.52	1204704.92	164374.29	19305.58	38589.02
2000	29.96	6897482.48	1270628.76	172190.34	24493.95	41095.74
2001	17.93	8134141.81	1699686.63	228557.88	29980.41	57196.61
2002	39.32	11332252.82	3875457.92	271026.11	36228.67	68807.96
2003	17.38	13301558.86	4161565.55	299224.96	44126.96	81008.74
2004	30.22	17321295.24	4419062.97	360802.97	56394.33	99004.04
2005	28.57 2	2269977.83	5372203.92	463420.03	67450.37	129258.10
2006	28.70	28662468.77	6723216.46	560246.06	80196.05	149639.23
2007	15.12	32995384.35	7654220.16	642276.42	91496.01	163988.81
2008	18.68	39157884.39	90396334.01	758839.77	108101.13	193750.28
2009	13.09	44285560.50	10419603.30	863402.42	121254.66	221181.95
2010	23.32	54612264.18	11683896.37	979564.05	135720.90	249711.48