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A new generator of probability distributions: the Normal–Kumaraswamy-G family of distributions

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Abstract. In a recent study, Osatohanmwen et al. (2019) developed the T - Kumaraswamy family of distributions which is a family of distributions with members having support on the unit interval. Five examples of such distributions were defined including the so-called 'normal - Kumaraswamy' distribution. A new method for generating probability distributions using the normal - Kumaraswamy distribution as the generator is presented in this paper. Some normal - Kumaraswamy generated distributions are presented alongside some general structural properties of the new generator. The new generator is in the same class as the beta-generated class and the Kumaraswamy-generated class which are well studied in the literature and it can compete favorably in modeling disparate data sets with these well-known and well-studied classes of distributions.

Keywords: Kumaraswamy distribution, normal distribution, probability distributions, parameters.

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1. Introduction

Methods for generating new continuous univariate probability distributions have continued to remain an active area of research in probability distribution theory within the last four decades. Azzalini (1985) introduced a system for generating new distributions by adding a skewing parameter to a symmetric distribution. Mudholkar and Srivastava (1993) defined the exponentiated class of distributions by exponentiating a given baseline distribution with a positive parameter. Eugene et al. (2002) defined the beta-generated class of distributions by using the beta distribution as the generator. Jones (2009) and Cordeiro and de Castro (2011) proposed the Kumaraswamy-generated class of distributions by using the Kumaraswamy distribution as the generator. More recently, Alzaatreh et al. (2014) defined the T–$R\{Y\}$ family of distributions as an extension of the beta and Kumaraswamy generated classes in other to include generators defined in some other interval other than the unit interval. Using the T–$R\{Y\}$ approach, Osatohanmwen et al. (2019) defined a new generalized family of distributions on the unit interval called the T – Kumaraswamy family of distributions. General properties of the generalized family were studied by the authors and five new members of the family were defined namely: the Weibull – Kumaraswamy {exponential}, log logistic – Kumaraswamy{exponential}, logistic – Kumaraswamy {extreme value}, exponential –Kumaraswamy {log logistic} and the normal – Kumaraswamy {logistic} distributions. The normal – Kumaraswamy {logistic} distribution was observed upon application to real data sets to be the most flexible among the five members, and also out-performed the beta and Kumaraswamy distributions in applications. The cumulative distribution function (cdf) and probability density function (pdf) of the normal – Kumaraswamy {logistic} distribution were given respectively by

$$F_X(x) = \Phi\left[\log\left((1-x^{\alpha})^{-\beta}-1\right)\right], \tag{1}$$

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$$f_X(x) = \frac{\alpha\beta\left(1-x^{\alpha}\right)^{-\beta-1}x^{\alpha-1}}{\left(1-x^{\alpha}\right)^{-\beta}-1}\phi\left[\log\left(\left(1-x^{\alpha}\right)^{-\beta}-1\right)\right], \tag{2}$$

$$\phi(.) = \Phi'(.), \qquad 0<x<1, \alpha, \beta>0,$$

where Φ is the cdf of the normal distribution.

In this paper, we define a new family of distributions generated by the normal – Kumaraswamy {logistic} distribution called the normal – Kumaraswamy – G family of distributions. The new family is defined in section 2 and some special members of the family are contained in section 3. General statistical properties of the new family are presented in section 4 and the maximum likelihood method of estimation of the parameters of the family is presented in section 5. Section 6 contains application and discussion of results and the paper concludes in section 7 with conclusion.

2. The Normal – Kumaraswamy – G family of distributions

Suppose $G(x;\Theta)$ is a valid cdf with parameter vector Θ a random variable X is said to follow the normal – Kumaraswamy – G (NKUM-G) family of distributions if it has the cdf

$$\begin{aligned} F(x;\alpha,\beta,\Theta) =& \alpha\beta\int_0^{G(x;\Theta)}\frac{\left(1-t^{\alpha}\right)^{-\beta-1}t^{\alpha-1}}{\left(1-t^{\alpha}\right)^{-\beta}-1}\phi\left(\log\left(\left(1-t\right)^{-\beta}-1\right)\right)dt \\ =& \Phi\left(\log\left(\left(1-G^{\alpha}(x;\Theta)\right)^{-\beta}-1\right)\right)\end{aligned} \tag{3}$$

$$\phi(.) = \Phi'(.), \qquad x\in \mathrm{R}, \Theta\in\mathrm{R}, \alpha, \qquad \beta>0.$$

The corresponding pdf of the NKUM - G family is given by

$$\begin{aligned} f(x;\alpha,\beta,\Theta) =& \frac{\alpha\beta g(x;\Theta)G^{\alpha-1}(x;\Theta)\left(1-G^{\alpha}(x;\Theta)\right)^{-\beta-1}}{\left(1-G^{\alpha}(x;\Theta)\right)^{-\beta}-1}\times \\ & \phi\left(\log\left(\left(1-G^{\alpha}(x;\Theta)\right)^{-\beta}-1\right)\right)\end{aligned} \tag{4}$$

$$\phi(.) = \Phi'(.), g(x;\Theta) = G'(x;\Theta), \qquad x\in\mathrm{R}, \qquad \Theta\in\mathrm{R}, \qquad \alpha,\beta>0.$$

The corresponding hazard function of the NKUM - G family is given by

$$\begin{aligned} h(x;\alpha,\beta,\Theta) =& \frac{\alpha\beta g(x;\Theta)G^{\alpha-1}(x;\Theta)\left(1-G^{\alpha}(x;\Theta)\right)^{-\beta-1}}{\left(\left(1-G^{\alpha}(x;\Theta)\right)^{-\beta}-1\right)\left(1-\Phi\left(\log\left(\left(1-G^{\alpha}(x;\Theta)\right)^{-\beta}-1\right)\right)\right)}\times \\ & \phi\left(\log\left(\left(1-G^{\alpha}(x;\Theta)\right)^{-\beta}-1\right)\right)\end{aligned} \tag{5}$$

$$\phi(.) = \Phi'(.), g(x;\Theta) = G'(x;\Theta), \qquad x\epsilon\mathrm{R}, \qquad \Theta\epsilon\mathrm{R}, \qquad \alpha,\beta>0.$$

3. Some members of the Normal – Kumaraswamy – G family of distributions

Here we present three members of the NKUM – G family namely: NKUM – Weibull, NKUM – normal and NKUM – Gumbel distributions. These new distributions are readily obtained by taking $G(x;\Theta)$ to be the cdf of the Weibull, normal and Gumbel distributions respectively.

3.1 *The Normal – Kumaraswamy – Weibull (NKUM – Weibull) distribution*

Suppose $G(x;\Theta)$ is the Weibull distribution with cdf $G(x;c,k) = 1 - \mathrm{e}^{-(x/c)^k}$, $c, k > 0$, the cdf of the NKUM – Weibull distribution is given by

$$F(x;\alpha,\beta,c,k) = \Phi\left(\log\left(\left(1-\left(1-\mathrm{e}^{-(x/c)^k}\right)^{\alpha}\right)^{-\beta} - 1\right)\right) \tag{6}$$

$$x > 0, \qquad \alpha,\beta,c,k > 0.$$

The corresponding pdf and hazard function of the NKUM – Weibull distribution are given respectively by

$$\begin{aligned} f(x;\alpha,\beta,c,k) = & \frac{\alpha\beta(x/c)^{k-1}\mathrm{e}^{-(x/c)^k}\left(1-\mathrm{e}^{-(x/c)^k}\right)^{\alpha-1}\left(1-\left(1-\mathrm{e}^{-(x/c)^k}\right)^{\alpha}\right)^{-\beta-1}}{c\left(1-\left(1-\mathrm{e}^{-(x/c)^k}\right)^{\alpha}\right)^{-\beta} - c} \times \\ & \phi\left(\log\left(\left(1-\left(1-\mathrm{e}^{-(x/c)^k}\right)^{\alpha}\right)^{-\beta} - 1\right)\right) \end{aligned} \tag{7}$$

$$x > 0, \qquad \alpha,\beta,c,k > 0,$$

$$\begin{aligned} h(x;\alpha,\beta,c,k) = & \frac{\alpha\beta(x/c)^{k-1}\mathrm{e}^{-(x/c)^k}\left(1-\mathrm{e}^{-(x/c)^k}\right)^{\alpha-1}\left(1-\left(1-\mathrm{e}^{-(x/c)^k}\right)^{\alpha}\right)^{-\beta-1}}{\left(c\left(1-\left(1-\mathrm{e}^{-(x/c)^k}\right)^{\alpha}\right)^{-\beta} - c\right)\left(1-\Phi\left(\log\left(\left(1-\left(1-\mathrm{e}^{-(x/c)^k}\right)^{\alpha}\right)^{-\beta} - 1\right)\right)\right)} \times \\ & \phi\left(\log\left(\left(1-\left(1-\mathrm{e}^{-(x/c)^k}\right)^{\alpha}\right)^{-\beta} - 1\right)\right) \end{aligned} \tag{8}$$

$$x > 0, \qquad \alpha,\beta,c,k > 0,$$

The density and the hazard plots of the NKUM – Weibull are shown in Figures 1 and 2 respectively.

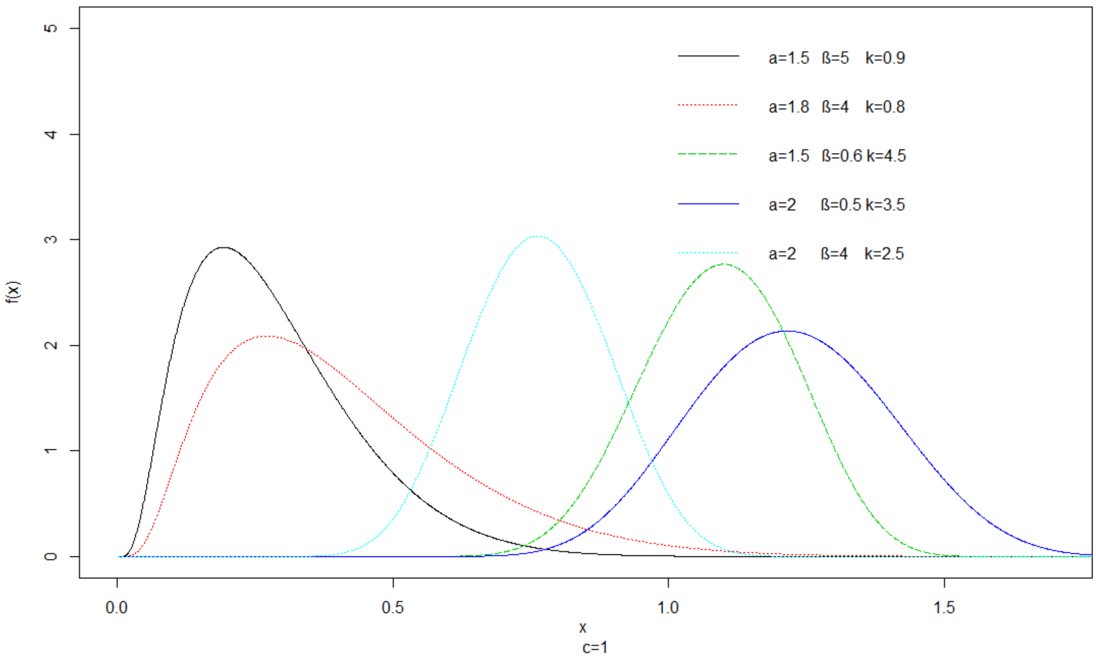

Figure 1: NKUM - Weibull density

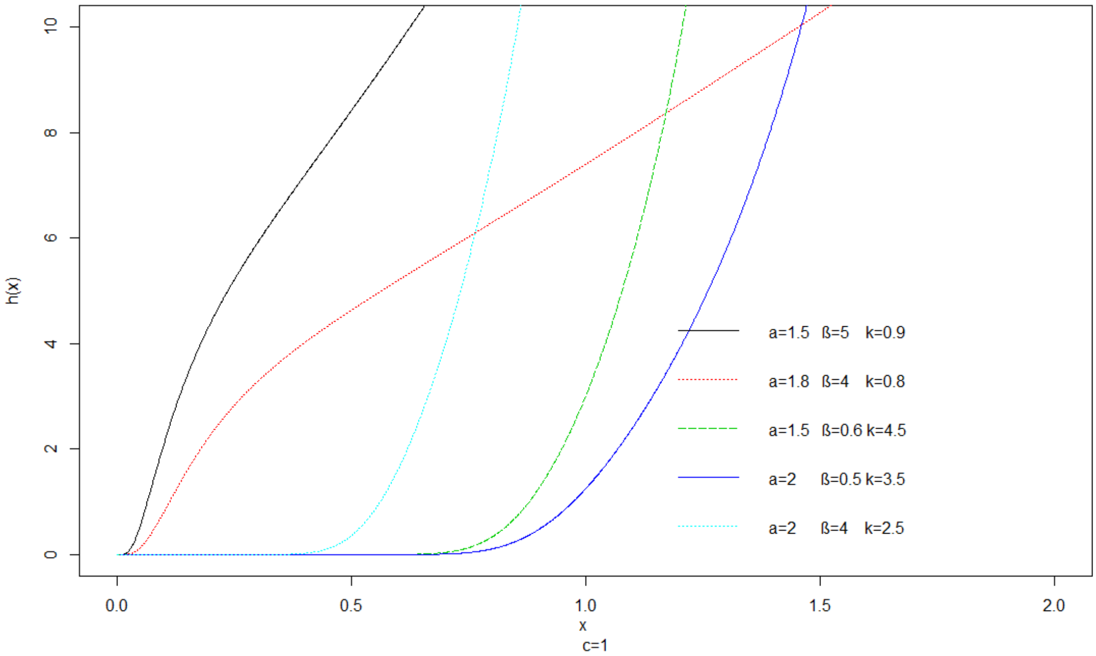

Figure 2: NKUM - Weibull hazard

Figure 1 shows that the NKUM – Weibull density can be right-skewed, left-skewed and almost symmetric. The hazard function of the distribution can be observed to be increasing from Figure 2.

3.2 *The Normal - Kumaraswamy - normal (NKUM - normal) distribution*

Taking $G(x;\Theta)$ to be the normal distribution with cdf $\Phi(z)$, the cdf of the NKUM – normal distribution is expressed as

$$F(z;\alpha,\beta) = \Phi\left(\log\left((1-\Phi^{\alpha}(z))^{-\beta}-1\right)\right), \tag{9}$$

$$z = (x-k)/c, -\infty < x, k < \infty, \alpha, \beta, c > 0.$$

The pdf and the hazard function of the NKUM – normal distribution are given respectively by

$$f(z;\alpha,\beta) = \frac{\alpha\beta\phi(z)\,\Phi^{\alpha-1}(z)\,(1-\Phi^{\alpha}(z))^{-\beta-1}}{c\,(1-\Phi^{\alpha}(z))^{-\beta}-c}\phi\left(\log\left((1-\Phi^{\alpha}(z))^{-\beta}-1\right)\right), \tag{10}$$

$$\phi(.) = \Phi'(.), \qquad z = (x-k)/c, \qquad -\infty < x, k < \infty, \qquad \alpha, \beta, c > 0,$$

$$\begin{aligned} h(z;\alpha,\beta) =& \frac{\alpha\beta\phi(z)\,\Phi^{\alpha-1}(z)\,(1-\Phi^{\alpha}(z))^{-\beta-1}}{\left(c\,(1-\Phi^{\alpha}(z))^{-\beta}-c\right)\left(1-\Phi\left(\log\left((1-\Phi^{\alpha}(z))^{-\beta}-1\right)\right)\right)}\times \\ & \phi\left(\log\left((1-\Phi^{\alpha}(z))^{-\beta}-1\right)\right), \end{aligned} \tag{11}$$

$$\phi(.) = \Phi'(.), z = (x-k)/c, -\infty < x, k < \infty, \alpha, \beta, c > 0.$$

The shapes of the density and hazard of the NKUM – normal distribution for $k = 0, c = 1$ are shown in Figures 3 and 4 respectively.

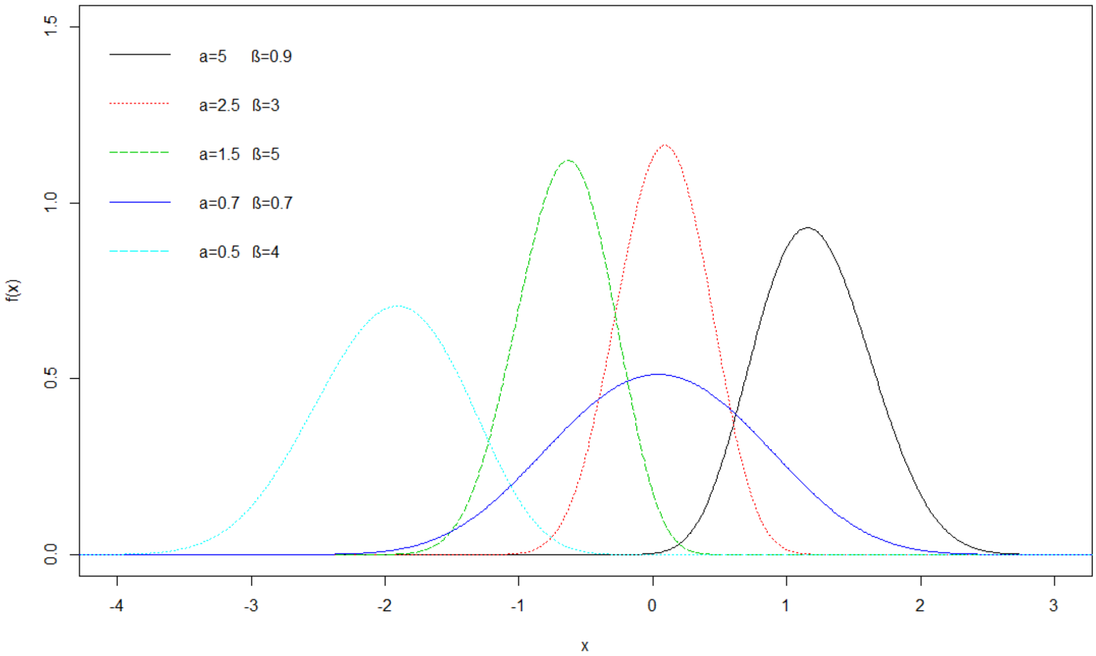

Figure 3: NKUM - normal density

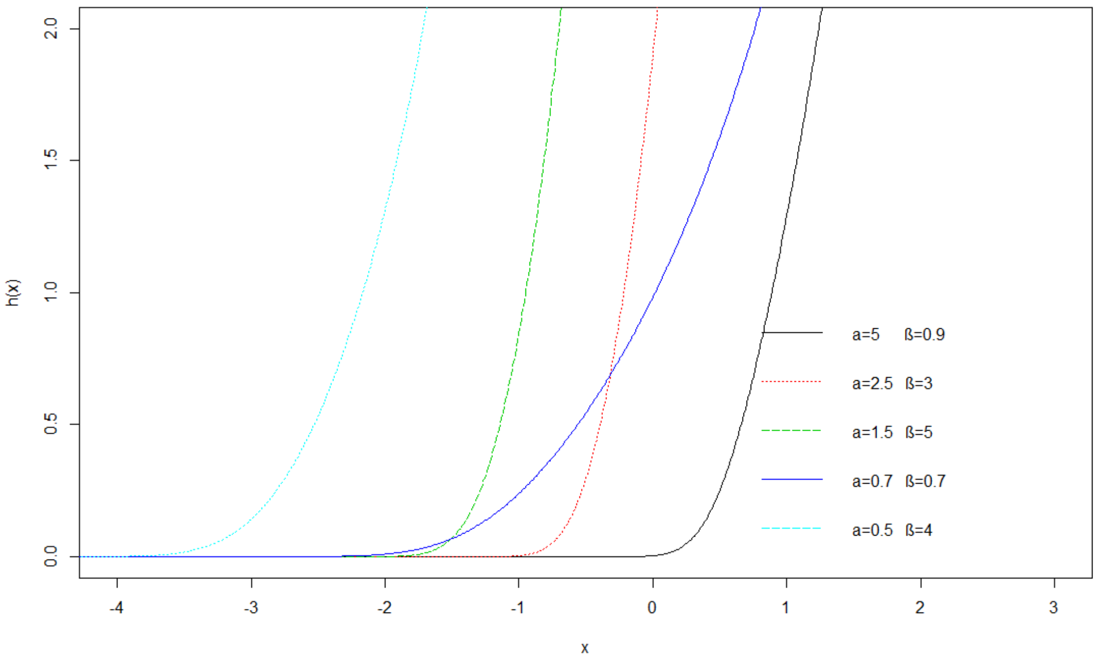

Figure 4: NKUM - normal hazard

Figure 3 shows that the graph of the NKUM – normal density can be skewed to the right and left thus adding skewness to the conventional symmetry of the normal distribution. The hazard of the distribution as shown in Figure 4 clearly shows that the distribution has an increasing hazard function.

3.3 *The Normal – Kumaraswamy (NKUM – Gumbel) distribution*

If $G(x;\Theta)$ is the Gumbel distribution with cdf $G(x;c,k)=\mathrm{e}^{-e^{-(x-k)/c}}, c>0, -\infty<k,x<\infty$, the cdf of the NKUM - Gumbel distribution is given by

$$F(x;\alpha,\beta,c,k)=\Phi\left(\log\left(\left(1-\left(\mathrm{e}^{-e^{-(x-k)/c}}\right)^{\alpha}\right)^{-\beta}-1\right)\right), \tag{12}$$

$$-\infty<x,k<\infty, \alpha,\beta,c>0.$$

The density and hazard functions of the NKUM – Gumbel distribution are given respectively by

$$\begin{aligned}f(x;\alpha,\beta,c,k)=&\frac{\alpha\beta\mathrm{e}^{-(x-k)/c}\left(\mathrm{e}^{-e^{-(x-k)/c}}\right)^{\alpha}\left(1-\left(\mathrm{e}^{-e^{-(x-k)/c}}\right)^{\alpha}\right)^{-\beta-1}}{c\left(1-\left(\mathrm{e}^{-e^{-(x-k)/c}}\right)^{\alpha}\right)^{-\beta}-c}\times\\&\phi\left(\log\left(\left(1-\left(\mathrm{e}^{-e^{-(x-k)/c}}\right)^{\alpha}\right)^{-\beta}-1\right)\right),\end{aligned} \tag{13}$$

$$-\infty<x,k<\infty, \alpha,\beta,c>0,$$

$$h(x;\alpha,\beta,c,k)=\frac{\alpha\beta\mathrm{e}^{-(x-k)/c}\left(\mathrm{e}^{-e^{-(x-k)/c}}\right)^{\alpha}\left(1-\left(\mathrm{e}^{-e^{-(x-k)/c}}\right)^{\alpha}\right)^{-\beta-1}}{\left(c\left(1-\left(\mathrm{e}^{-e^{-(x-k)/c}}\right)^{\alpha}\right)^{-\beta}-c\right)\left(1-\Phi\left(\log\left(\left(1-\left(\mathrm{e}^{-e^{-(x-k)/c}}\right)^{\alpha}\right)^{-\beta}-1\right)\right)\right)}\times$$

$$\phi\left(\log\left(\left(1-\left(\mathrm{e}^{-e^{-(x-k)/c}}\right)^{\alpha}\right)^{-\beta}-1\right)\right), \tag{14}$$

$$-\infty<x,k<\infty, \alpha,\beta,c>0.$$

The density and the hazard of the NKUM – Gumbel are shown in Figures 5 and 6 respectively.

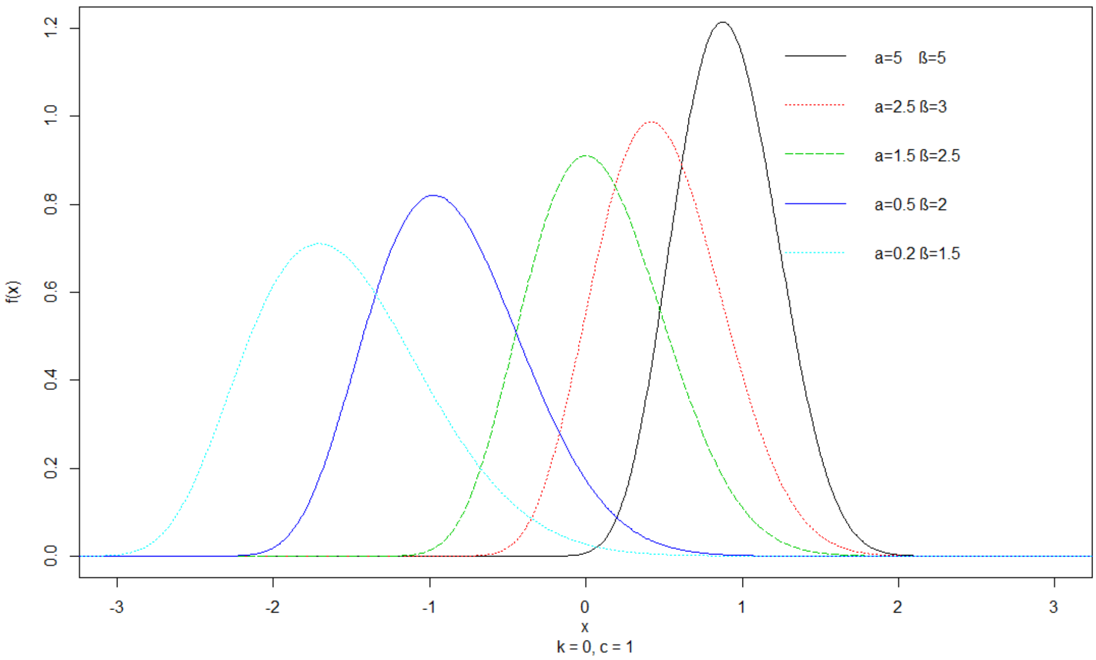

Figure 5: NKUM - Gumbel density

Figure 5 reveals that the NKUM – Gumbel density can be skewed to the right, left and also symmetric. The hazard function of the distribution as shown in Figure 6 shows increasing property.

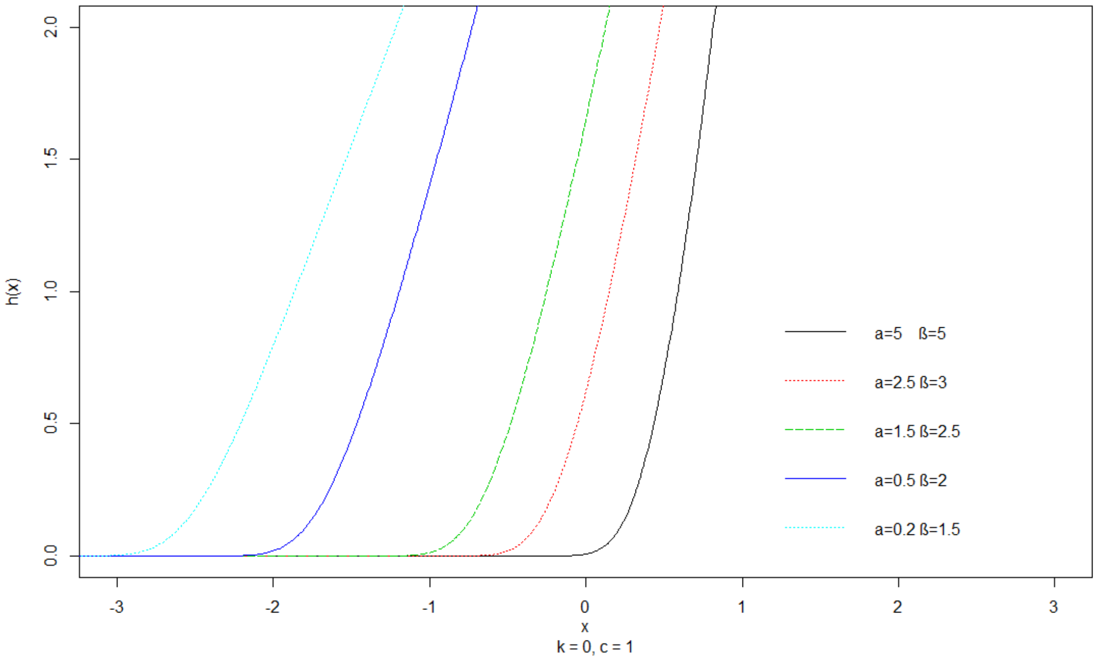

Figure 6: NKUM - Gumbel hazard

4. General properties of the Normal – Kumaraswamy – G family of distributions

Here some general properties of the NKUM – G family are presented. These properties include the quantiles, moments and order statistics. We begin by finding a useful expansion for the cdf and pdf of the NKUM – G family.

4.1 *A useful expansion for the family*

Using the following results in Gradshteyn and Ryzhik (2000):

$$\log(x) = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1}, x > 0,$$

$$\operatorname{erf}(x) = \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^m x^{2n+1}}{(2n+1)\, n!}, \pi = 3.141593, x\epsilon\mathrm{R},$$

and the fact that

$$\Phi(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right),$$

As well as the results of the generalized binomial theorem, the cdf of the NKUM – G family can be represented by

$$F(x;\alpha,\beta,\Theta) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=0}^{\infty} w_n \kappa_{\delta_{k_s},p,q,m,n} G^q(x;\Theta), \tag{15}$$

where

$$w_n = \frac{(-1)^n \, 2^{-(2n+1)/2}}{(2n+1)\, n!},$$

$$\kappa_{\delta_{k_s},p,q,m,n} = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} ... \sum_{k_{2n+1}=1}^{\infty} \delta\left(k_1, k_1, ..., k_{2n+1}\right) \psi_{p,q,m,n},$$

$$\delta\left(k_1, k_1, ..., k_{2n+1}\right) = \delta_{k_1} \delta_{k_2} ... \delta_{k_{2n+1}},$$

$$\delta_{k_s} = \sum_{j=0}^{2k_s-1} \sum_{m=0}^{\infty} (-1)^{j+m} \, 2^{j+1} \binom{2k_s-1}{j} \binom{\beta j}{m}, s = 1, 2, ..., 2n+1,$$

$$\psi_{p,q,m,n} = \sum_{p=0}^{\infty} \sum_{q=0}^{p} (-1)^{p+q} \binom{\alpha m(2n+1)}{p} \binom{p}{q},$$

$$\binom{z}{w} = \frac{z(z-1)(z-2)...(z-w+1)}{w!}.$$

Differentiating (15) w.r.t. x gives the pdf of the NKUM – G in expanded form as

$$f\left(x; \alpha, \beta, \Theta\right) = \frac{1}{\pi} \sum_{n=0}^{\infty} w_n \kappa_{\delta_{k_s},p,q,m,n} q g(x;\Theta) G^{q-1}\left(x;\Theta\right). \tag{16}$$

It is evident from (16) that the density of the NKUM – G is an infinite linear combination of the density of the exponentiated family of distribution with exponentiation parameter q (Mudholkar and Srivastava (1993). Thus some structural properties of the NKUM – G family of distributions can be obtained from well- established properties of the exponentiated family of distributions. The properties of the exponentiated family of distributions have been studied by many authors and profoundly by Gupta et al. (1998).

4.2 *Quantile function of the family*

The quantile function of the NKUM – G family, obtained from finding the root of the equation $F(Q(p); \alpha, \beta, \Theta) = p$, is expressed as

$$Q(p) = G^{-1}\left\{\left[1 - \left(e^{\Phi^{-1}(p)} + 1\right)^{-1/\beta}\right]^{1/\alpha}\right\}. \tag{17}$$

Suppose U is a uniform random variable on $(0, 1)$, random variates can be obtained from the NKUM – G family using the relation

$$X = G^{-1}\left\{\left[1 - \left(e^{\Phi^{-1}(U)} + 1\right)^{-1/\beta}\right]^{1/\alpha}\right\}. \tag{18}$$

4.3 *Moments of the family*

Suppose Y is a random variable following the exponentiated family of distributions with exponentiation parameter q, let $E(Y_q^r)$ be the r^{th} non-central moment of Y then the r^{th} non-central moment of the NKUM – G family random variable X can be expressed as

$$E(X^r) = \frac{1}{\pi} \sum_{n=0}^{\infty} w_n \kappa_{\delta_{k_s},p,q,m,n} E(Y_q^r). \tag{19}$$

Since the quantity in (16) is absolutely integrable the incomplete moment $I_X(t)$ and the moment generating function$M_X(t)$ of the NKUM – G random variable X can be written as

$$I_X(t) = \int_{-\infty}^{t} x^s f(x; \alpha, \beta, \Theta) dx = \frac{1}{\pi} \sum_{n=0}^{\infty} w_n \kappa_{\delta_{k_s},p,q,m,n} I_Y(t),$$

where $I_X(t)$ is the incomplete moment of the random variable Y following the exponentiated family of distribution with exponentiation parameter q and

$$M_X(t) = \frac{1}{\pi} \sum_{n=0}^{\infty} w_n \kappa_{\delta_{k_s},p,q,m,n} E(\mathrm{e}^{tY}), t \epsilon \mathrm{R},$$

where $E(e^{tY})$ is the moment generating function of the random variable Y following the exponentiated family of distribution with exponentiation parameter q.

4.4 *Order statistics*

Order statistics play a very crucial row in non-parametric statistics and inference. For a random sample $X_1, X_2, ..., X_n$ of size n from the NKUM – G distribution with corresponding order statistics $X_{1:n} < X_{2:n} < ... < X_{n:n}$, the pdf of the k^{th} order statistic is given by

$$f_{k:n}(x) = \frac{1}{\mathrm{B}(k, n-k+1)} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k},$$

and then

$$f_{k:n}(x) = \sum_{i=0}^{n-k} \frac{(-1)^i}{\mathrm{B}(k, n-k+1)} \binom{n-k}{i} f(x)[F(x)]^{k+i-1}, \tag{20}$$

where $f(x)$ and $F(x)$ are taken to be the pdf and cdf of the NKUM - G distribution respectively.

5. Maximum likelihood estimation of the parameters of the family

For a complete random sample $x_1, x_2, ..., x_n$ of size n from the NKUM – G distribution, the total log-likelihood function is given by

$$L = n\log\alpha + n\log\beta + \sum_{i=1}^{n} \log(g(x_i;\Theta)) + (\alpha-1)\sum_{i=1}^{n} \log(G(x_i;\Theta)) - (\beta+1)\sum_{i=1}^{n} \log(1-G^{\alpha}(x_i;\Theta))$$

$$+ \sum_{i=1}^{n} \log[\phi(\log((1-G^{\alpha}(x_i;\Theta))^{-\beta}-1))] - \sum_{i=1}^{n} \log((1-G^{\alpha}(x_i;\Theta))^{-\beta}-1). \tag{21}$$

The maximum likelihood estimates of the parameters α, β and the vector of parameters Θ are obtained by numerically solving the systems of equations

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha}\sum_{i=1}^{n}\log(G(x_i;\Theta)) + (\beta+1)\sum_{i=1}^{n}\frac{G^{\alpha}(x_i;\Theta)\log(G(x_i;\Theta))}{1-G^{\alpha}(x_i;\Theta)}$$

$$-\beta\sum_{i=1}^{n}\frac{G^{\alpha}(x_i;\Theta)\log(G(x_i;\Theta))(1-G^{\alpha}(x_i;\Theta))^{-\beta-1}}{(1-G^{\alpha}(x_i;\Theta))^{-\beta}-1}$$

$$+\beta\sum_{i=1}^{n}\frac{G^{\alpha}(x_i;\Theta)\log(G(x_i;\Theta))(1-G^{\alpha}(x_i;\Theta))^{-\beta-1}\phi'(\log((1-G^{\alpha}(x_i;\Theta))^{-\beta}-1))}{((1-G^{\alpha}(x_i;\Theta))^{-\beta}-1)\phi(\log((1-G^{\alpha}(x_i;\Theta))^{-\beta}-1))} = 0$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log(1-G^{\alpha}(x_i;\Theta)) + \sum_{i=1}^{n}\frac{(1-G^{\alpha}(x_i;\Theta))^{-\beta}\log(1-G^{\alpha}(x_i;\Theta))}{(1-G^{\alpha}(x_i;\Theta))^{-\beta}-1}$$

$$-\sum_{i=1}^{n}\frac{(1-G^{\alpha}(x_i;\Theta))^{-\beta}\log(1-G^{\alpha}(x_i;\Theta))\phi'(\log((1-G^{\alpha}(x_i;\Theta))^{-\beta}-1))}{((1-G^{\alpha}(x_i;\Theta))^{-\beta}-1)\phi(\log((1-G^{\alpha}(x_i;\Theta))^{-\beta}-1))} = 0$$

$$\frac{\partial L}{\partial \Theta_k} = \sum_{i=1}^{n}\frac{\partial(\log(g(x_i;\Theta))}{\partial \Theta_k} + (\alpha-1)\sum_{i=1}^{n}\frac{\partial(\log(G(x_i;\Theta))}{\partial \Theta_k} - (\beta+1)\sum_{i=1}^{n}\frac{\partial(\log(1-G^{\alpha}(x_i;\Theta))}{\partial \Theta_k}$$

$$+\sum_{i=1}^{n}\left(\frac{\partial\log[\phi(\log((1-G^{\alpha}(x_i;\Theta))^{-\beta}-1))]}{\partial \Theta_k} - \frac{\partial\log((1-G^{\alpha}(x_i;\Theta))^{-\beta}-1)}{\partial \Theta_k}\right) = 0$$

for the parameters α, β and the vector of parameters Θ. Since the resulting systems of equations are not in closed form, the solutions can be found numerically using some specialized numerical iterative scheme such as the Newton-Raphson type algorithms, which can be implemented on several computing software like R, SAS, MATHEMATICA and MATLAB.

6. Application and discussion of results

In this section, some members of the NKUM – G family of distributions namely: NKUM – Weibull (NKUM – W), NKUM – normal (NKUM – N) and NKUM – Gumbel (NKUM – Gu) are applied for the fitting of a real data set and the result compared with those of some members of the beta – generated and Kumaraswamy – generated classes namely: beta - normal (BN) (Eugene et al. 2002), beta - weibull (BW) (Famoye et al. 2005), beta - Gumbel (BG) (Nadarajah and Kotz, 2004), Kumaraswamy – Normal (Kw-N), Kumaraswamy – Weibull (Kw-W) and Kumaraswamy – Gumbel (Kw-G) (Cordeiro and de Castro 2011) distributions. The data set represents the breaking stress of carbon fibers of 50 mm length (GPa). The data was obtained from Nicholas and Padgett (2006). The data set is unimodal, approximately symmetric and almost mesokurtic (Skewness = -0.1315 and kurtosis = 0.2231). The data is presented in Table 1.

Table 1: Breaking stress of carbon fibers of 50 mm length (GPa)

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38,4.42,4.70,4.90

The results of the fit of all the nine (9) distributions to the data set using the maximum likelihood method are contained in Table 2. These results include the estimates of the parameters and their standard errors (in parenthesis) of all the fitted distributions, the Kolmogorov – Smirnov (K-S) statistic and its corresponding p-value (in parenthesis), Akaike Information Criterion (AIC) and the loglikelihood (loglik) values of all the fitted distributions. The graph of the fitted densities alongside the histogram of the data set is given by Figure 7.

Results in Table 2 show that all the nine (9) distributions provided good fits to the data set with the Kw – G distribution having the overall best fit since it has the smallest AIC value, smallest K – S statistic value and the highest p-value among all the fitted distributions. However, members of the NKUM – G family of distributions performed so well in fitting the data set. In fact, all the members of the NKUM – G family outperformed the Kw – W and Kw – N in fitting the data set. This application clearly reveals that probability distributions belonging to the NKUM – G family can compete favorably with the well-established members of the beta-generated and Kumaraswamy generated classes of distributions and hence present another useful alternative to these generators of flexible probability distributions.

Table 2: Maximum likelihood estimates of parameters of the fitted distributions to the data

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{k}$	loglik	AIC	K - S
NKUM - Gu	38.8132	26.7777	4.8078	-8.5712	-85.85	179.71	0.0832
-	(120.11)	(64.641)	(3.0154)	(16.162)	-	-	(0.7193)
NKUM - W	0.2252	0.0701	1.1674	2.4583	-85.79	179.58	0.0921
-	(0.0214)	(0.0057)	(0.0041)	(0.0040)	-	-	(0.5981)
NKUM - N	1.0285	1.5279	1.7038	3.3041	-85.88	179.77	0.0808
-	(8.3330)	(10.330)	(7.7239)	(3.5105)	-	-	(0.7511)
BG	0.6545	14.2410	2.1458	5.5606	-85.42	178.84	0.0728
-	(0.9943)	(20.437)	(1.5486)	(1.7371)	-	-	(0.8502)
BW	0.7629	3.9685	4.6600	4.0583	-85.92	179.84	0.0811
-	(0.3774)	(43.581)	(12.947)	(1.3049)	-	-	(0.7468)
BN	2.8032	6.9044	2.0008	3.9804	-85.45	178.91	0.0735
-	(31.978)	(82.030)	(12.390)	(7.4269)	-	-	(0.8425)
Kw - G	4.5765	23.0857	2.7284	2.0617	-85.42	178.83	0.0727
-	(19.279)	(33.302)	(1.0785)	(11.449)	-	-	(0.8513)
Kw - W	0.7907	5.2449	5.1496	4.1787	-86.03	180.05	0.1453
-	(0.9002)	(144.77)	(39.864)	(6.3870)	-	-	(0.1115)
Kw - N	1.4149	0.0999	0.4657	1.0992	-88.78	185.56	0.1597
-	(0.0172)	(0.0123)	(0.0020)	(0.0020)	-	-	(0.0614)

7. Conclusion

A new family of univariate probability distributions has been introduced in this paper. The new family of distributions can be expressed as infinite linear combination of the exponentiated family of distributions widely studied in the literature. This result makes it possible to study the properties of the new family by simply using the results obtained for the exponentiated family of distributions. Three members of the new family have been defined, and the shapes of their density and hazard functions shown. The flexibility of members of the new family in real life applications have been shown by application to a real data set where the members of the new family not only provided adequate fit to the data set, but also competed favorably with members of the beta and Kumaraswamy generated classes and hence, presents good alternatives to these well-established distributions. It is hoped that several other members of the new family are studied and applied to disparate data sets.

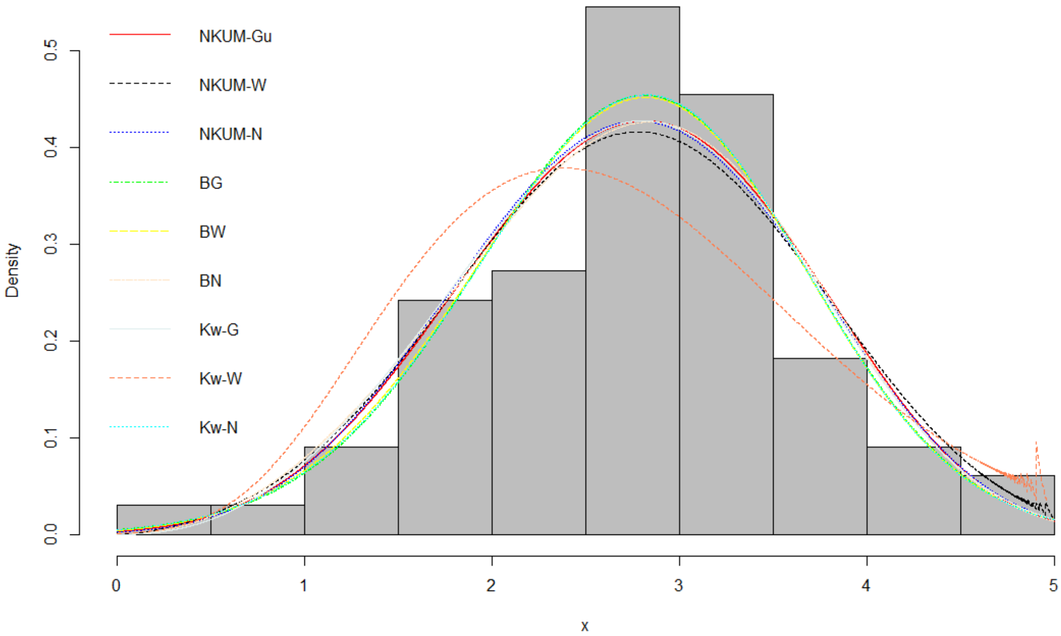

Figure 7: Fitted density for breaking stress of carbon fibers of 50 mm Length (GPa)

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