BENIN JOURNAL OF STATISTICS
ISSN 2682-5767
Vol. 4, pp. 135– 153 (2021)

Some Results on the Transmuted Odd Lindley-Rayleigh Distribution

S. A. Umar[1*], A. B. Bukar[2], M. S. Makama[3] and T. G. Ieren[4]

[1]*Department of Policy, Planning and Research, National Space Research and Development Agency, Abuja, Nigeria*; [2]*Department of Mathematics, Nigeria Defence Academy, Kaduna*; [3]*Department of Medical Statistics, Shehu Idris Collage of Health Sciences and Technology, Makarfi, Kaduna*; [4]*Department of Statistics and Operations Research, MAUTech, P. M. B. 2076, Yola, Nigeria*

(*Received: 22 September 2020; accepted: 23 January 2021*)

Abstract. In this article, a new distribution named "Transmuted odd Lindley-Rayleigh Distribution (TOLRD)" is presented. The respective definitions of its probability density function (pdf) and cumulative distribution function (cdf) using the Quadratic Rank Transmutation Map proposed by shaw et al. Some properties of the new distribution such as moments, moment generating function, characteristics function, quantile function, survival function, hazard function and order statistics were studied. The estimation of the distribution?s parameters was conducted using the method of maximum likelihood. The performance of the proposed probability distribution, TOLRD is compared to some other generalizations of Rayleigh distribution using one real life dataset and two simulated datasets. The results obtained were compared using the values of the AIC, CAIC, BIC and HQIC of the fitted distributions and it showed that the TOLRD outperforms all the other fitted distributions.

Keywords: Lindley distribution, Rayleigh distribution, odd Lindley Rayleigh distribution, hazard rate function, order statistics, reliability function.

Published by: Department of Statistics, University of Benin, Nigeria

1. Introduction

The Rayleigh distribution was obtained from the amplitude of sound resulting from many important sources by Rayleigh (1980). It is a continuous probability

*Corresponding author. Email: omarson07@yahoo.com

distribution with a wide range of applications such as in life testing experiments, reliability analysis, applied statistics and clinical studies. It is a special case of the two parameter Weibull distribution when the shape parameter takes a value of 2. Its origin and other important characteristics can be found in the work of Siddiqui (1962), Hirano (1986) as well as Howlader and Hossian (1995).

The probability density function (pdf) and the cumulative distribution function (cdf) of the Rayleigh distribution with scale parameter θ are respectively given as:

$$g(x) = \theta x e^{-\frac{\theta}{2}x^2} \tag{1}$$

And

$$G(x) = 1 - e^{-\frac{\theta}{2}x^2} \tag{2}$$

where $x > 0, \theta > 0$. Recently, several authors have proposed efficient families of probability distributions and it has been proven that they lead to more flexible probability models. These proposed families among others include the quadratic rank transmutation map proposed by Shaw and Buckley (2007), the Weibull-X family of distribution by Alzaatreh et al. (2013), the Weibull-G family of distributions by Bourguignon et al. (2014), the Gamma-X family by Alzaatreh et al. (2014), a Lomax-G family by Cordeiro et al. (2014), a new Weibull-G family by Tahir et al. (2016), a Lindley-G family by Cakmakyapan and Ozel (2016), a Gompertz-G family by Alizadeh et al. (2017), an odd Lindley-G family by Gomes-Silva et al. (2017) and an odd Lomax generator of distributions by Cordeiro et al. (2019).

Using these families and other methods have led to many extensions of the Rayleigh distribution some of which are; the generalized Rayleigh distribution by Kundu et al. (2005), Bivariate generalized Rayleigh distribution by Abdel-Hady (2013), Transmuted Rayleigh distribution by Merovci (2013), the Weibull-Rayleigh distribution by Merovci and Eltabal (2015), generalized Weibull-Rayleigh distribution by Yahaya and Alaku (2018), Weibull-Rayleigh distribution by Merovci and Elbatal (2015), transmuted Weibull-Rayleigh distribution by Yahaya et al. (2020), the Transmuted Inverse Rayleigh distribution studied by Ahmad et al., (2014) and the odd Lindley-Rayleigh distribution by Ieren et al. (2020).

Apart from the extensions of the Rayleigh distribution, other competitive models arising from the proposed families of distributions include the odd Lindley inverse exponential distribution by Ieren and Abdullahi (2020a), the transmuted normal distribution by Ieren and Abdullahi (2020b), the Weibull-Exponential distribution by Oguntunde et al. (2015), the transmuted Weibull-exponential distribution by Yahaya and Ieren (2017), the Weibull-Frechet distribution by Afify et al. (2016), the transmuted Lomax distribution by Ashour and Eltehiwy

(2013), the Gompertz-Lindley distribution by Koleoso et al. (2019), and many others.

Inspired by the above listed families and the related extended probability distributions, this study will propose another extension of the Rayleigh distribution by using the Quadratic Rank Transmutation Map proposed by Shaw and Buckley (2007), this proposed flexible distribution is referred to as "the transmuted odd Linley-Rayleigh distribution (TOLRD)".

DEFINITION 1.1 *(Ieren et al., 2020) A random variable X is said to have the Odd Lindley-Rayleigh distribution if its cdf and pdf are respectively given (3) and (4) as:*

$$G(x) = 1 - \frac{\alpha e^{-\frac{\theta}{2}x^2} + 1}{\alpha + 1} e^{-\alpha(e^{\frac{\theta}{2}x^2} - 1)} \tag{3}$$

and

$$g(x) = \frac{\alpha^2 \theta x e^{-\frac{\theta}{2}x^2} + 1}{1 + \alpha} e^{-\alpha[e^{\frac{\theta}{2}x^2} - 1]} \tag{4}$$

where $x > 0\alpha > 0$ and $\theta > 0$ and θ and α are the scale and shape parameter of the Odd Lindley-Rayleigh distribution respectively.

2. Transmuted Odd Lindley Rayleigh Distribution

DEFINITION 2.1 *A random variable X is said to have a transmuted distribution function proposed by Shaw and Buckley (2007), if its pdf and cdf are respectively given by;*

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)] \tag{5}$$

and

$$F(x) = (1 + \lambda)G(x) - \lambda[g(x)]^2 \tag{6}$$

where $x > 0, -1 \leq \lambda \leq 1$ is the transmuted parameter, G(x) is the cdf of any continuous distribution and f(x) and g(x) are the associated pdf of F(x) and G(x) respectively.

Using equation (3) and (4) in (5) and (6) and simplifying, the cdf and pdf of the transmuted Odd Lindley-Rayleigh distribution are given as:

$$F(x) = (1+\lambda)\left(1 - \frac{\alpha e^{-\frac{\theta}{2}x^2}+1}{\alpha+1}e^{-\alpha(e^{\frac{\theta}{2}x^2}-1)}\right) - \lambda\left(1 - \frac{\alpha e^{-\frac{\theta}{2}x^2}+1}{\alpha+1}\times\right.$$

$$e^{-\alpha(e^{\frac{\theta}{2}x^2}-1)2} \tag{7}$$

and

$$f(x) = \frac{\alpha^2\theta x e^{-\frac{\theta}{2}x^2}+1}{1+\alpha}e^{-\alpha[e^{\frac{\theta}{2}x^2}-1]}\left[1+\lambda-2\lambda\left(1-\frac{\alpha e^{-\frac{\theta}{2}x^2}+1}{(\alpha+1)}e^{-\alpha(e^{\frac{\theta}{2}x^2}-1)}\right)\right] \tag{8}$$

respectively, where $x > 0, \alpha > 0, \theta > 0, -1 \leq \lambda \leq 1 and \alpha$ and θ are the scale and shape parameter respectively while is called the transmuted parameter of the transmuted odd Lindley-Rayleigh distribution. Hence, equation (7) and (8) are the cdf and pdf of the transmuted odd Lindley-Rayleigh distribution (TOLRD).

Graphical representation of the pdf and cdf of the transmuted odd Lindley-Rayleigh distribution. Given some values for the parameters θ, α and λ. The plot

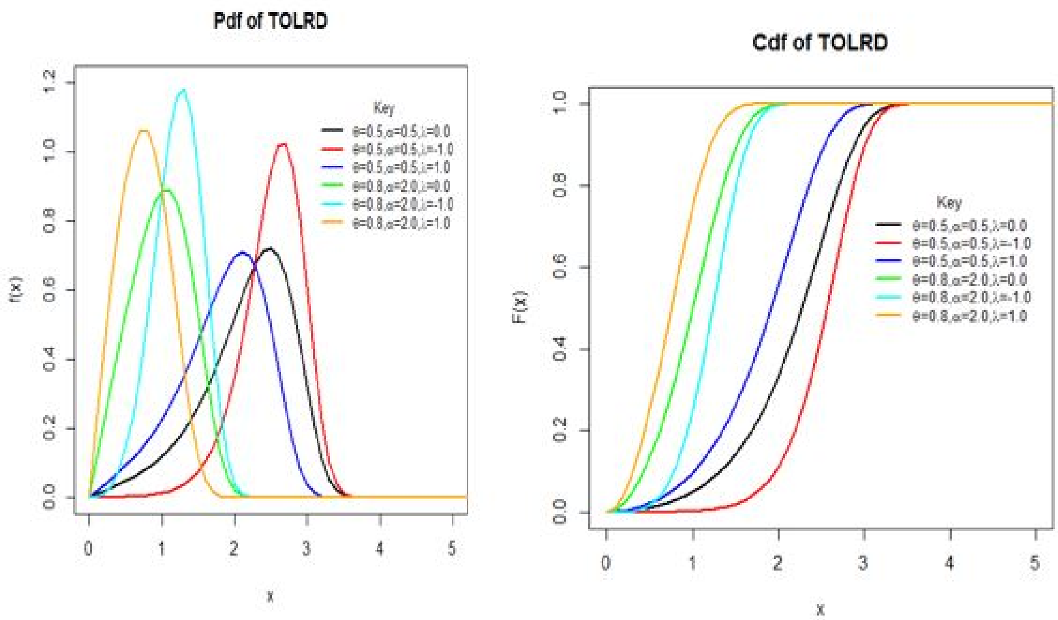

Figure 1: PDF and CDF of the TOLRD for some selected values of $\theta, \alpha and \lambda$.

for the pdf shows that the TOLRD is skewed with various shapes and therefore will be a good model for various kinds of datasets.

2.1 *Mathematical and statistical properties of TOLRD*

2.1.1 Moments

Let X be a continuous random variable; the n^{th} moment of X is defined as:

$$\mu_n = E(X^n) = \int_0^{\infty} x^n f(x) dx \tag{9}$$

where $f(x)$ is the pdf of the transmuted odd Lindley-Rayleigh distribution is as given in equation (8) as:

$$f(x) = \frac{\alpha^2(1-\lambda)\theta x e^{2\frac{\theta}{2}x^2}}{(1+\alpha)} e^{-\alpha\left(\frac{\theta}{2}x^2-1\right)} + \frac{2\lambda\alpha^2\theta x e^{2\frac{\theta}{2}x^2}}{(1+\alpha)^2}\left(\alpha e^{\frac{\theta}{2}x^2+1}\right) e^{-2\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)} \tag{10}$$

Before substitution in (9), we perform the expansion and simplification of the pdf as follows. First, by expanding the exponential term in (10) using power series, we obtain:

$$e^{-\alpha\left(\frac{\theta}{2}x^2-1\right)} = \sum_{i=1}^{\infty} \frac{(-\alpha)^i}{i!}\left(e^{\frac{\theta}{2}x^2-1}\right)^i \tag{11}$$

Making use of the result in (11) above, equation (10) becomes

$$f(x) = \frac{\alpha^2(1-\lambda)\theta x e^{2\frac{\theta}{2}x^2}}{(1+\alpha)} \sum_{i=0}^{\infty} \frac{(-\alpha)^i}{i!}\left(e^{\frac{\theta}{2}x^2-1}\right)^i + \frac{2\lambda\alpha^2\theta x e^{2\frac{\theta}{2}x^2}}{(1+\alpha)^2}\left(\alpha e^{\frac{\theta}{2}x^2}+1\right)$$
$$\sum_{i=0}^{\infty} \frac{(-2\alpha)^2}{i!}\left(e^{\frac{\theta}{2}x^2}-1\right)^i \tag{12}$$

Also, using the generalized binomial theorem, we can write the last term from the above result as:

$$\left((e^{\frac{\theta}{2}x^2}-1)^i\right) = \sum_{j=0}^{\infty}(-1)^j e^{\frac{\theta}{2}(i-j)x^2} \tag{13}$$

$$\mu_n^{'} = \frac{\alpha^2\theta}{(1+\alpha)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\alpha)^i(-1)^{i+j}2^{\frac{n}{2}+1}}{(\theta(2+i-j))^{\frac{\pi}{2}+1} i!}\binom{1}{j}\Gamma\left(\frac{n+2}{2}\right)$$
$$\left[((1-\lambda)+\frac{2\lambda(2)}{(1+\alpha)}\left(\frac{\alpha(\theta(2+i-j))^{-\frac{(T+1)}{2}}}{(\theta(3+i-j))^{\frac{\pi}{2}+1}}+1\right)\right] \tag{14}$$

The mean

The mean of the TOLRD can be obtained from the nth moment of the distribution when n=1 as follows:

$$\mu_1^{'} = \frac{\alpha^2\theta}{(1+\alpha)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\alpha)^i(-1)^{i+j}2^{\frac{3}{2}}}{(\theta(2+i-j))^{\frac{3}{2}}i!}\binom{i}{j}\Gamma\left(\frac{3}{2}\right)$$
$$\left[(1-\lambda)+\frac{2\lambda(2)^i}{(1+\alpha)}\left(\frac{\alpha(\theta(2+i-j))^{-(\frac{3}{2})}}{(\theta(3+i-j))^{\frac{3}{2}}}+1\right)\right] \quad (15)$$

Also the second moment of the TOLRD is obtained from the n^th moment of the distribution when $n=2$ as

$$\mu_2^{'} = \frac{4\alpha^2\theta}{(1+\alpha)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\alpha)^i(-1)^{i+j}}{(\theta(2+i-j))^{2}i!}\binom{i}{j}\left[(1-\lambda)+\frac{2\lambda(2)^i}{(1+\alpha)}\times\right.$$
$$\left.\left(\frac{\alpha(\theta(2+i-j))^{-2}}{(\theta(3+i-j))^{2}}+1\right)\right] \quad (16)$$

The variance

The n^{th} central moment or moment about the mean of X, say μ_n and variance $Var(x)$ can be obtained as

$$\mu_n = E(X-\mu_1)^n = \sum_{i=0}^{n}\binom{i}{\mu}'i_1\mu_{n-i}^{'} \quad (17)$$

$$Var(X) = \left(\frac{2}{\theta(j-2-i)}\right)^2\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\alpha^{2+i}\theta}{i!(1+\alpha)}\binom{i}{j}-\right.$$
$$\left.\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\alpha^{2+i}\theta}{i!(1+\alpha)}\binom{i}{j}\Gamma(\frac{3}{2})\right\}^2\right\} \quad (18)$$

2.1.2 Moment generating function

The mgf of a random variable X can be obtained by

$$M_x(t) = E[e^{tx}] = \int_0^\infty e^{tx} f(x)dx \tag{19}$$

$$M_x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left\{ \frac{\alpha^2\theta}{(1+\alpha)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha)^i (-1)^{i+j} 2^{\frac{k+2}{2}}}{(\theta(2+i-j))^{\frac{k+2}{2}} i!} \binom{i}{j} \Gamma\left(\frac{k+2}{2}\right) \right. \\ \left[(1-\lambda) + \frac{2\lambda(2)^i}{(1+\alpha)} \left(\frac{\alpha(\theta(2+i-j))^{-\frac{k+2}{2}}}{(\theta(3+i-j))^{\frac{k+2}{2}}} + 1 \right) \right] \tag{20}$$

2.1.3 Characteristics function

The characteristics function of a random variable X is defined by:

$$\varphi_x(t) = E(e^{itx}) = \int_0^\infty e^{itx} f(x)dx \tag{21}$$

$$\varphi_x(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \left[\frac{\alpha^2\theta}{(1+\alpha)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha)^i (-1)^{i+j} 2^{\frac{k-1}{2}}}{(\theta(2+i-j))^{\frac{k\alpha}{2}} i!} \binom{i}{j} \Gamma\left(\frac{k+2}{2}\right) \right] \\ \left[(1-\lambda) + \frac{2\lambda(2)}{(1+\alpha)} \left(\frac{\alpha(\theta(2+i-j))^{-\left(\frac{\langle 2}{2}\right)}}{(\theta(3+i-j))^{\frac{k-2}{2}}} + 1 \right) \right] \tag{22}$$

2.1.4 Reliability analysis

The survival function $S(x)$ and Hazard function $H(x)$ of the TOLRD are given by:

$$S(x) = \frac{\alpha \mathrm{e}^{\frac{\theta}{2}x^2} + 1}{(\alpha+1)} \mathrm{e}^{-\alpha\left(e^{\frac{\theta^2}{2}-1}\right)} \left[1 - \lambda + \lambda \frac{\alpha \mathrm{e}^{\frac{\theta}{2}x^2} + 1}{(\alpha+1)} \mathrm{e}^{-\alpha\left(e^{\frac{\partial}{2}^2-1}\right)} \right] \tag{23}$$

$$h(x) = \frac{\frac{\alpha^2\theta x e^{2\frac{\theta}{2}x^2}}{(1+\alpha)e^{\alpha\left(e^{\frac{\theta}{2}x^2-1}\right)}} \left[1 + \lambda - 2\lambda \left(1 - \frac{\alpha e^{\frac{\theta}{2}x^2}+1}{(\alpha+1)} e^{-\alpha\left(\frac{\theta}{2}x^2-1\right)} \right) \right]}{\frac{\alpha e^{\frac{\theta}{2}x^2}+1}{(\alpha+1)} e^{-\alpha\left(\frac{\theta}{2}x^2-1\right)} \left[1 - \lambda + \lambda \frac{\alpha e^{\frac{\theta}{2}x^2}+1}{(\alpha+1)} e^{-\alpha\left(\frac{\theta}{2}x^2-1\right)} \right]} = \frac{\alpha^2\theta x e^{\frac{\theta}{2}x^2}}{\alpha e^{\frac{\theta}{2}x^2+1}} \tag{24}$$

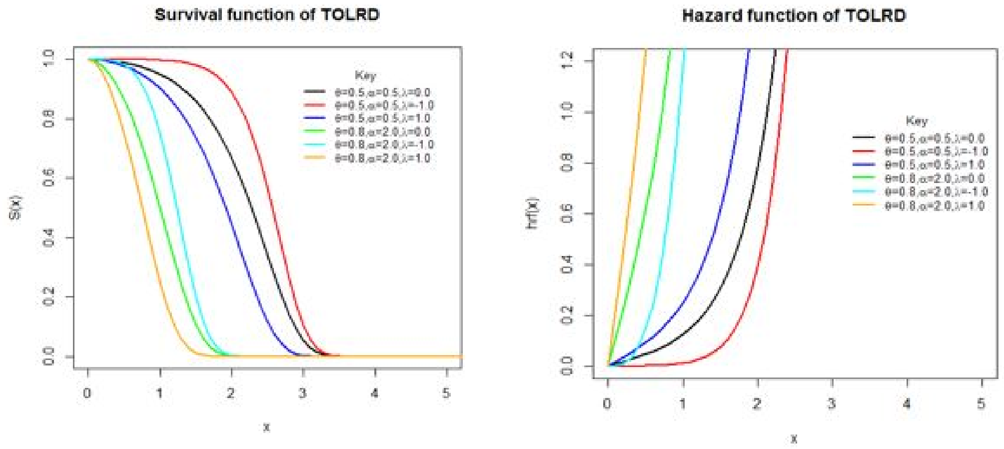

Figure 2: Survival function of the TOLRD at different parameter values and Hazard function of the TOLRD at different parameter values.

2.1.5 Quantile function

Hyndman and Fan (1996) defined the quantile function for any distribution in the form $Q(u) = F^{-1}(u)$ where $Q(u)$ is the quantile function of $F(x)$ for $0 < u < 1$ using $F(x)$ to be the cdf of the TOLRD and inverting it as above will give us the quantile function as follows:

$$F(x) = (1+\lambda)\left(1 - \frac{\alpha e^{\frac{\theta}{2}x^2} + 1}{(\alpha+1)}\right)$$

$$F(x) = (1-\lambda)\left(1 - \tfrac{\alpha^2\theta x e^{\frac{\theta}{2}x^2}}{(\alpha+1)} e^{-\alpha(\frac{\theta}{2}x^2-1)}\right) - \lambda\left((1 - \tfrac{\alpha^2\theta x e^{\frac{\theta}{2}x^2}}{(\alpha+1)} e^{-\alpha(\frac{\theta}{2}x^2-1)})\right)^2 = u$$

$$Q(u)\times \sqrt{\frac{2}{\theta}\log\left\{-\frac{1}{\alpha}\left[W_{-1}\left(-(\alpha+1)\left(1 - \frac{(1+\lambda) - \sqrt{(1-\lambda)^2 - 2\lambda u}}{2\lambda}\right)e^{-(\alpha+1)}\right) + 1\right]\right\}} \quad (25)$$

Similarly, random numbers can be simulated from the TOLRD by setting $Q(u) = X$ and this process is called simulation using inverse transformation method. This means for any $\alpha, \theta > 0$ and $u \in (0, 1)$:

$$X(u)\times \sqrt{\frac{2}{\theta}\log\left\{-\frac{1}{\alpha}\left[W_{-1}\left(-(\alpha+1)\left(1 - \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}\right)e^{-(\alpha+1)}\right) + 1\right]\right\}} \quad (26)$$

where u is a uniform variate on the unit interval $(0, 1)$ and W_{-1} represents the negative branch of the Lambert function.

2.1.6 Skewness and kurtosis

The quantile based measures of skewness and kurtosis will have employed due to non-existence of the classical measures in some cases. The Bowley's measure of skewness (Kennedy and Keeping, 1962.) based on quartiles is given by;

$$SK = \frac{Q(\frac{3}{4}) - 2Q(\frac{1}{2}) + Q(\frac{1}{4})}{Q(\frac{3}{4}) - Q(\frac{3}{4})} \tag{27}$$

And the Moores (1998) kurtosis is on octiles and is given by;

$$KT = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) - Q(\frac{3}{8}) + Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{1}{4})} \tag{28}$$

2.1.7 Order statistics

Suppose $X_1, X_2, X_3, \cdots\cdots, X_n$ is a random sample from a distribution with pdf, f(x), and let $X_{1;n}, X_{2;n}, X_{3;n}, \cdots\cdots, X_{n;n}$ denote the corresponding order statistic obtained from this sample. The pdf, $f_{i;n}(x)$ of the i^{th} order statistic can be defined as;

$$f_{i;n}(x) = \frac{n!}{(i-1)(n-i)!} f(x)F(x)^{i-1}[1-F(x)]^{n-1} \tag{29}$$

where $f(x)$ and $F(x)$ are the pdf and cdf of the proposed distribution respectively. Hence, the pdf of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the TOLRD are given by;

$$f_{i;n}(x) = n\sum_{k=o}^{n-i}\binom{n-i}{k}\left[\frac{\alpha^2\theta x e^{2\frac{\theta}{2}x^2}}{(1+\alpha)e^{\alpha(e^{\frac{\theta}{2}x^2})}}\left[1+\lambda-2\lambda\left(1-\frac{(\alpha e^{\frac{\theta}{2}x^2+1})}{(\alpha+1)e^{\alpha(\frac{\theta}{2}x^2-1)}}\right)\right]\right]$$
$$\left[1-\left\{\frac{\alpha e^{2\frac{\theta}{2}x^2}+1}{(\alpha+1)e^{\alpha(e^{\frac{\theta}{2}x^2})}}\left(1-2\lambda\frac{\alpha e^{2\frac{\theta}{2}x^2}+1}{(\alpha+1)e^{\alpha(e^{\frac{\theta}{2}x^2})}}\right)\right\}\right] \tag{30}$$

and

$$f_{n;n}(x) = n\left[\frac{\alpha^2\theta x e^{2\frac{\theta}{2}x^2}}{(1+\alpha)e^{\alpha(e^{\frac{\theta}{2}x^2})}}\left[1+\lambda-2\lambda\left(1-\frac{(\alpha e^{\frac{\theta}{2}x^2+1})}{(\alpha+1)e^{\alpha(\frac{\theta}{2}x^2-1)}}\right)\right]\right]$$
$$\left[1-\left\{\frac{\alpha e^{2\frac{\theta}{2}x^2}+1}{(\alpha+1)e^{\alpha(e^{\frac{\theta}{2}x^2})}}\left(1-2\lambda\frac{\alpha e^{2\frac{\theta}{2}x^2}+1}{(\alpha+1)e^{\alpha(e^{\frac{\theta}{2}x^2})}}\right)\right\}\right] \tag{31}$$

respectively.

2.1.8 Estimation of parameters of TOLRD using maximum likelihood method

Let $X_1, ..., X_n$ be a sample of size $'n'$ independently and identically distributed random variables from the $TOLRD$ with unknown parameters $\theta, \alpha, and \lambda$ defined previously. The likelihood function obtained from the pdf is given by;

$$L\left(X|\alpha,\theta,\lambda\right) = \frac{\alpha^{2n}\theta^{n}x_i\,\mathrm{e}^{2\frac{\theta}{2}\sum\limits_{i=1}^{n}x_i^2}}{\left(1+\alpha\right)^{n}}\,\mathrm{e}^{-\alpha\sum\limits_{i=1}^{n}\left(e^{\frac{\theta}{2}x_i^2}-1\right)}\times$$

$$\prod_{i=1}^{n}\left[1-\lambda+2\lambda\frac{\alpha\,\mathrm{e}^{\frac{\theta}{2}x_i^2}+1}{\left(\alpha+1\right)}\,\mathrm{e}^{-\alpha\left(\mathrm{e}^{\frac{\theta}{2}x_i^2}-1\right)}\right]$$

Let the log-likelihood function, $l = \log L\left(X|\alpha,\theta,\lambda\right)$, therefore,

$$l = 2n\log\alpha + n\log\theta - n\log\left(1+\alpha\right) + \sum_{i=1}^{n}x_i + \theta\sum_{i=1}^{n}x_i^2 - \alpha\sum_{i=1}^{n}\left(e^{\frac{\theta}{2}x_i^2}-1\right)$$
$$+\sum_{i=1}^{n}\log\left[1-\lambda+2\lambda\frac{\alpha\,\mathrm{e}^{\frac{\theta}{2}x_i^2}+1}{\left(\alpha+1\right)}\,\mathrm{e}^{-\alpha\left(\mathrm{e}^{\frac{\theta}{2}x_i^2}-1\right)}\right] \quad (32)$$

Differentiating l partially with respect to θ , α and λ respectively gives;

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}x_i^2 - \frac{\alpha}{2}\sum_{i=1}^{n}x_i^2\,\mathrm{e}^{\frac{\theta}{2}x_i^2} - \frac{\alpha^2\lambda}{\left(\alpha+1\right)}\times$$
$$\sum_{i=1}^{n}\left\{\frac{x_i^2\,\mathrm{e}^{\frac{\theta}{2}x_i^2}\,\mathrm{e}^{-\alpha\left(\mathrm{e}^{\frac{\theta}{2}x_i^2}-1\right)+\frac{\theta}{2}x_i^2}}{\left[1-\lambda+2\lambda\frac{\alpha\,\mathrm{e}^{\frac{\theta}{2}x_i^2}+1}{(\alpha+1)}\,\mathrm{e}^{-\alpha\left(\mathrm{e}^{\frac{\theta}{2}x_i^2}-1\right)}\right]}\right\} \quad (33)$$

$$\frac{\partial l}{\partial\alpha} = \frac{2n}{\alpha} - \frac{n}{\left(\alpha+1\right)} - \sum_{i=1}^{n}\left(\mathrm{e}^{\frac{\theta}{2}x_i^2}-1\right) + \frac{2\lambda}{\left(\alpha+1\right)^2}$$
$$\sum_{i=1}^{n}\left\{\frac{\left(\mathrm{e}^{\frac{\theta}{2}x_i^2}-1\right)\left(\alpha\left(\alpha+1\right)\mathrm{e}^{\frac{\theta}{2}x_i^2}+\alpha+2\right)}{\left[1-\lambda+2\lambda\frac{\alpha\,\mathrm{e}^{\frac{\theta}{2}x_i^2}+1}{(\alpha+1)}\,\mathrm{e}^{-\alpha\left(\mathrm{e}^{\frac{\theta}{2}x_i^2}-1\right)}\right]\mathrm{e}^{\alpha\left(\mathrm{e}^{\frac{\theta}{2}x_i^2}-1\right)}}\right\} \quad (34)$$

$$\frac{\partial l}{\partial \lambda} = \frac{2}{(\alpha+1)} \sum_{i=1}^{n} \left\{ \frac{\left(\alpha\, \mathrm{e}^{\frac{\theta}{2} x_i^2} + 1\right) \mathrm{e}^{-\alpha\left(\mathrm{e}^{\frac{\theta}{2} x_i^2} - 1\right)}}{\left[1 - \lambda + 2\lambda \frac{\alpha\, \mathrm{e}^{\frac{\theta}{2} x_i^2} + 1}{(\alpha+1)}\, \mathrm{e}^{-\alpha\left(\mathrm{e}^{\frac{\theta}{2} x_i^2} - 1\right)}\right]} \right\} \tag{35}$$

3. Result and Discussion

This section presents one real life dataset and two simulated datasets, their descriptive statistics and applications to some selected generalizations of the Rayleigh distribution. We have compared the fitness of the TOLRD to those of five generalizations of the Rayleigh model with the conventional Rayleigh distribution. The models are: odd Lindley-Rayleigh distribution (OLRD), generalized Weibull-Rayleigh distribution (GWRD), transmuted Weibull-Rayleigh distribution (TWRD), Weibull-Rayleigh distribution (WRD), transmuted Rayleigh distribution (TRD) and the conventional Rayleigh distribution (RD) and their performance is being evaluated and compare under this section using four information criteria.

To check performance of the models mentioned above, we made use of some criteria: the AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan Quin information criterion). The formulas for these statistics are given as follows:

$$AIC = 2ll + 2k + BIC = 2ll + klog(n), CAIC = -2ll + \frac{2kn}{(n-k-1)}$$

and

$$HQIC = -2ll + 2klog[log(n)]$$

where $\ell\ell$ denotes the log-likelihood function evaluated at the MLEs, k is the number of model parameters and n is the sample size. These results will be obtained in R statistical software using "AdequacyModel"package. The optimization algorithm used in this paper was based on the swarm intelligence and is called the Particle Swarm Optimization (PSO). This iterative method is found to be simple, efficient and robust. The PSO is a stochastic search method introduced by Nicholas and Padgett (2006) based on simple social behavior exhibited by birds and insects and, due to its simplicity in implementation it has gained great popularity in optimization. It also has high level of convergence and low computational cost compared with other heuristic search methods.

Decision benchmark

The model with the lowest values of these statistics would be chosen as the best model to fit the data. We also considered two simulated and one real life dataset for fitting the above selected models and the descriptive statistics for these data sets are also provided in Tables 3.1, 3.5 and 3.9 accordingly.

Data set I

This is a real life dataset and it represents the strength of 1.5cm glass fibers initially collected by members of staff at the UK national laboratory. It has been used by Afify and Aryal (2016), Barreto-Souza et al. (2011), Bourguignon et al. (2014), Oguntunde et al. (2015) as well as Smith and Naylor (1987). Its summary is given as follows:

Table 1: Summary statistics for the dataset I

Parameters	n	Min	Q_1	Median	Q_3	Mean	Max	Variance
Data set I	63	0.550	1.375	1.590	1.685	1.507	2.240	0.105
Parameters	Skewness	Kurtosis						
Data set I	-0.8786	3.9238						

Table 2: Maximum likelihood parameter estimates for dataset I

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
TOLRD	1.6233643	0.1603326	-	0.7547535
OLRD	1.0624889	0.5233132	-	-
GWRD	0.4704822	4.0026582	2.8069557	-
TWRD	0.1146162	7.6172655	0.9942883	-0.9899103
WRD	0.07109269	6.55035969	1.10263880	-
TRD	0.4426424	-	-	0.8594637
RD	0.842494	-	-	-

Table 3: The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset I

Distribution	$\hat{\ell}$	AIC	C AIC	BIC	HQIC	Ranks
TOLRD	13.81757	33.63514	34.04192	40.06455	36.16386	1^{st}
OLRD	18.38938	40.77875	40.97875	45.06502	42.46457	2^{nd}
GWRD	39.33739	84.67479	85.08157	91.10419	87.20351	3^{rd}
TWR	68.58643	145.1729	145.8625	153.7454	148.5445	4^{th}
WRD	80.17205	166.3441	166.7509	172.7735	168.8728	7^{th}
TRD	51.12219	106.2444	106.4444	110.5307	107.9302	5^{th}
RD	49.79089	101.5818	101.6474	103.7249	102.4247	6^{th}

Table 4: The $A^{*}, W^{*}, K-S$ statistic P-values for dataset I

Distribution	A^{*}	W^{*}	K-S	P-Value(K-S)	Ranks
TOLRD	0.6912041	0.1166265	0.11221	0.4058	1^{st}
OLRD	0.9370426	0.1658518	0.15934	0.08159	3^{rd}
TWRD	2.454438	0.4474747	0.28357	7.96e^{-05}	4^{th}
WRD	2.490174	0.4542347	0.44399	3.265e^{-11}	7^{th}
TRD	2.654657	0.4836042	0.32199	4.243 e^{-06}	5^{th}
RD	2.553825	0.4654581	0.3339	1.586e^{-06}	6^{th}

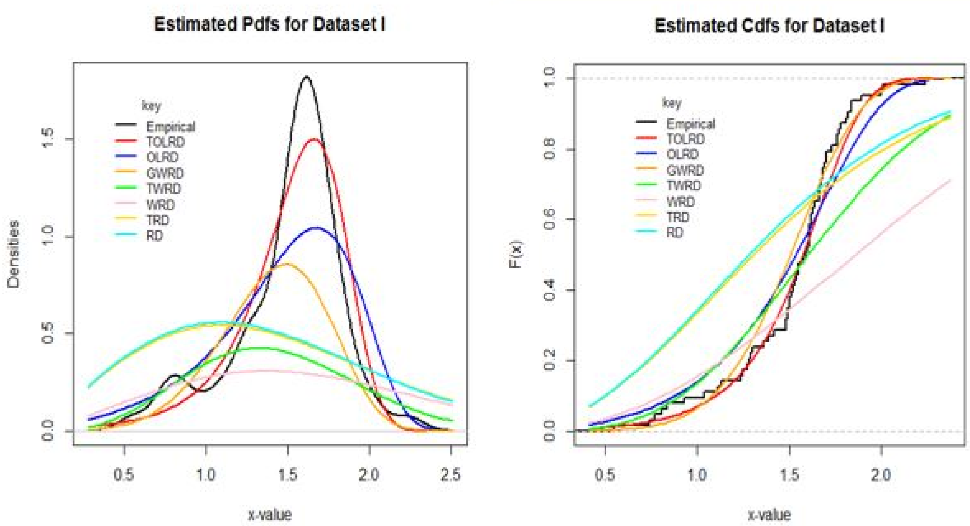

Figure 3: Plots of the estimated densities and cdfs of the TOLRD and other fitted distributions to dataset I.

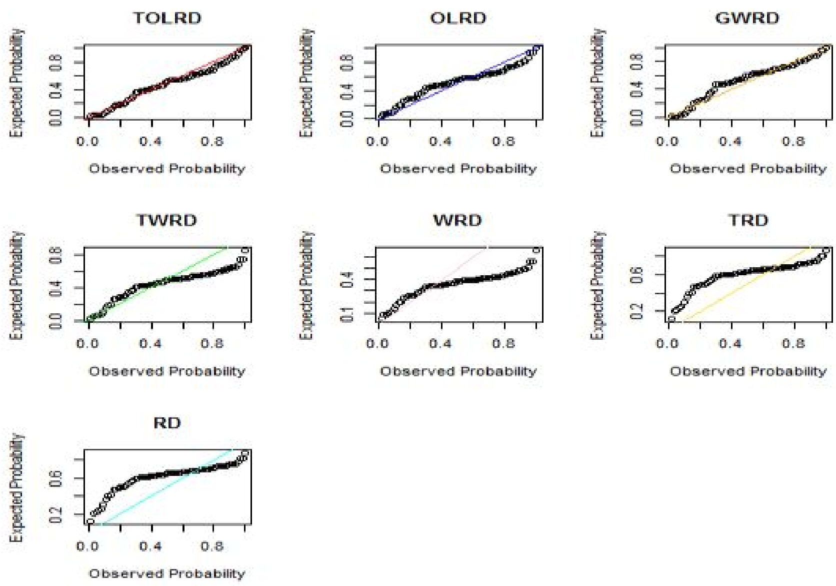

Figure 4: Probability plots for the fit of the TOLRD and other fitted models based on dataset I.

Data set II

This data set represents 25 identically and independently distributed random samples from the proposed distribution using its quantile function at some selected values of the parameters. Its summary is given as follows:

Table 5: Summary statistics for data set II

Parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum
Dataset II	25	0.1986	0.6426	0.8944	1.1645	0.8919	1.5240
Parameters	Variance	Skewness	Kurtosis				
Dataset II	0.1371	-0.01066	-1.0730				

Table 6: Maximum likelihood parameter estimates for dataset II

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
TOLRD	2.0726415	0.6023984	-	-0.1164494
OLRD	2.170051	0.526270	-	-
GWRD	1.239544	1.595929	2.005641	-
TWRD	0.03982647	7.68232810	0.77182676	0.91757273
WRD	0.1664427	7.5782168	1.1737399	-
TRD	0.8017181	-	-	0.9917666
RD	1.752578	-	-	-

Table 7: The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset II

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	Ranks
TOLRD	3.976447	13.95289	15.09575	17.60952	14.96709	2^{nd}
OLRD	3.975391	11.95078	12.49624	14.38853	12.62691	1^{st}
GWRD	4.795573	15.59115	16.734	19.24777	16.60534	3^{rd}
TWRD	19.36886	46.73772	48.73772	51.61322	48.08998	6^{th}
WRD	23.44288	52.88576	54.02862	56.54239	53.89996	7^{th}
TRD	12.21544	28.43087	28.97633	30.86863	29.107	5^{th}
RD	12.08046	26.16092	26.33483	27.3798	26.49898	4^{th}

Table 8: The $A^{*}, W^{*}, K-S$ statistic and P-values for dataset II

Distribution	A^{*}	W^{*}	K-S	P-Value(K-S)	Ranks
TOLRD	0.1933367	0.02730844	0.099304	0.9456	2^{nd}
OLRD	0.1930592	0.02725893	0.096254	0.9575	1^{st}
GWRD	0.4504254	0.06973345	0.10519	0.9179	3^{rd}
TWRD	0.8952867	0.1468559	0.27441	0.03728	6^{th}
WRD	0.6965211	0.1119639	0.32734	0.006916	7^{th}
TRD	0.7759702	0.1260532	0.22664	0.1307	5^{th}
RD	0.7639347	0.1239765	0.2586	0.05804	4^{th}

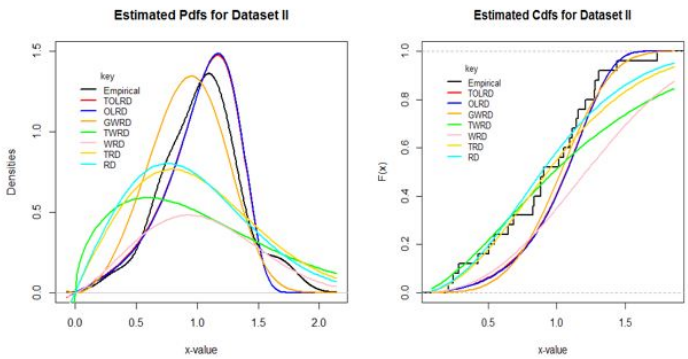

Figure 5: Plots of the estimated densities and cdfs of the TOLRD and other fitted distributions to dataset II.

Data set III:

This data set represents 75 random samples from the transmuted odd Lindley-Rayleigh distribution obtained by using the quantile function derived from the distribution. These sample values are summarized as follows:

From the descriptive statistics in tables 1,5 and 9 for the real life data and two simulated datasets respectively, we observed that the real life dataset is negatively skewed while the second and third simulated datasets are approximately

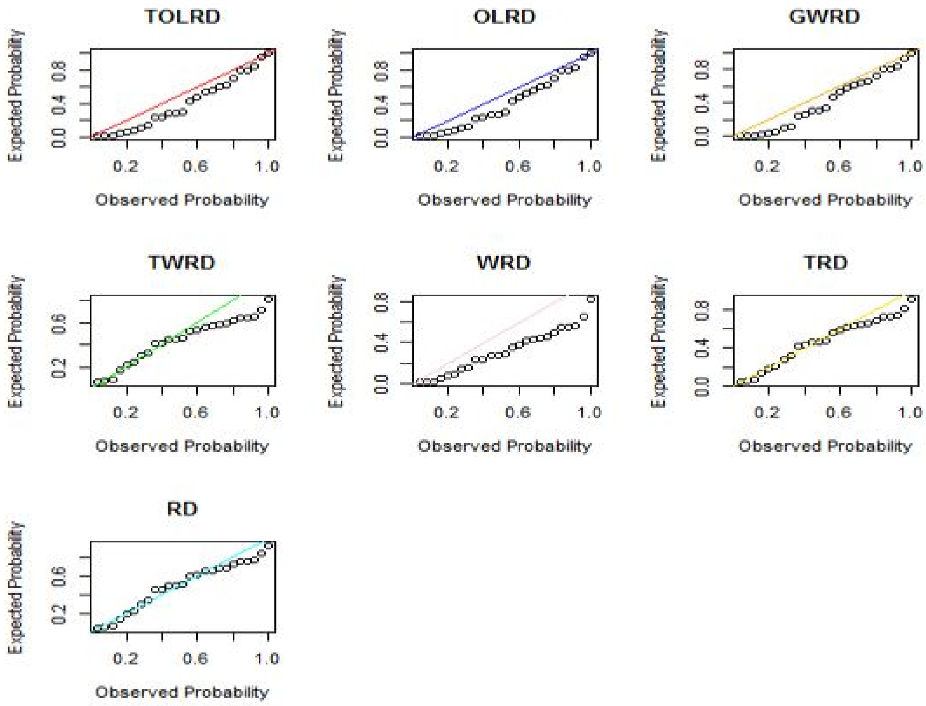

Figure 6: Probability plots for the fit of the TOLRD and other fitted models based on dataset II.

Table 9: Summary statistics for data set III

Parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum
Dataset III	75	0.2385	0.6042	1.0082	1.2316	0.9481	1.8208
Parameters	Variance	Skewness	Kurtosis				
Dataset III	0.14923	-0.0363	-0.8872				

Table 10: Maximum Likelihood Parameter Estimates for dataset II

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
TOLRD	0.9516180	1.7359889	-	0.0768925
OLRD	0.9303831	1.8491023	-	-
GWRD	0.7814648	3.0432610	1.3787909	-
TWRD	0.1375127	7.3534691	0.5437992	-0.8588687
WRD	0.1469494	6.2596086	0.8152597	-
TRD	0.9648429	-	-	0.9991048
RD	1.911893	-	-	-

normal. Tables 2, 6 and 3.10 list the values of the MLEs of the model parameters for all the datasets, whereas the values of AIC, CAIC, BIC and HQIC are listed in Tables 3, 7 and 11 for datasets I, II and III respectively. Also, the values of $A*$, W^* and K-S for datasets I, II and III are provided in Tables 4, 8 and 12 respectively.

The plots of the fitted TOLRD density and cumulative distribution functions with those of competing distributions for datasets I, II and III are displayed in Figures 3.1, 3.3 and 3.5 respectively. The PP-plots of the fitted distributions are also given in Figures 3.2, 3.4 and 3.6 for datasets I, II and III respectively. From the results based on all the measures above, it is observed that the

Table 11: The statistics $\hat{\ell}$, AIC, CAIC, BIC and HQIC for dataset III

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	Ranks
TOLRD	32.12251	70.24501	70.58304	77.19748	73.02106	1^{st}
OLRD	32.12017	68.24034	68.407	72.87531	70.09103	2^{nd}
GWRD	21.61369	49.22737	49.5654	56.17984	52.00342	3^{rd}
TWRD	66.97369	141.9474	142.5188	151.2173	145.6488	7^{th}
WRD	66.66516	139.3303	139.6683	146.2828	142.1064	6^{th}
TRD	38.0747	80.1494	80.31607	84.78438	82.0001	5^{th}
RD	38.06388	78.12775	78.18254	80.44524	79.0531	4^{th}

Table 12: The A^* statistic and P-values for dataset II

Distribution	A^*	W^*	K-S	P-Value(K-S)	Ranks
TOLRD	0.2799036	0.03782669	0.054497	0.9703	1^{st}
OLRD	0.27794	0.03718699	0.056006	0.9623	2^{nd}
GWRD	0.5705062	0.09489576	0.080078	0.6917	3^{rd}
TWRD	1.087207	0.1888586	0.24685	0.0001642	7^{th}
WRD	0.8483612	0.1466537	0.16572	0.0287	6^{th}
TRD	0.7918959	0.1355383	0.13348	0.126	5^{th}
RD	0.7924078	0.1356459	0.13047	0.1423	4^{th}

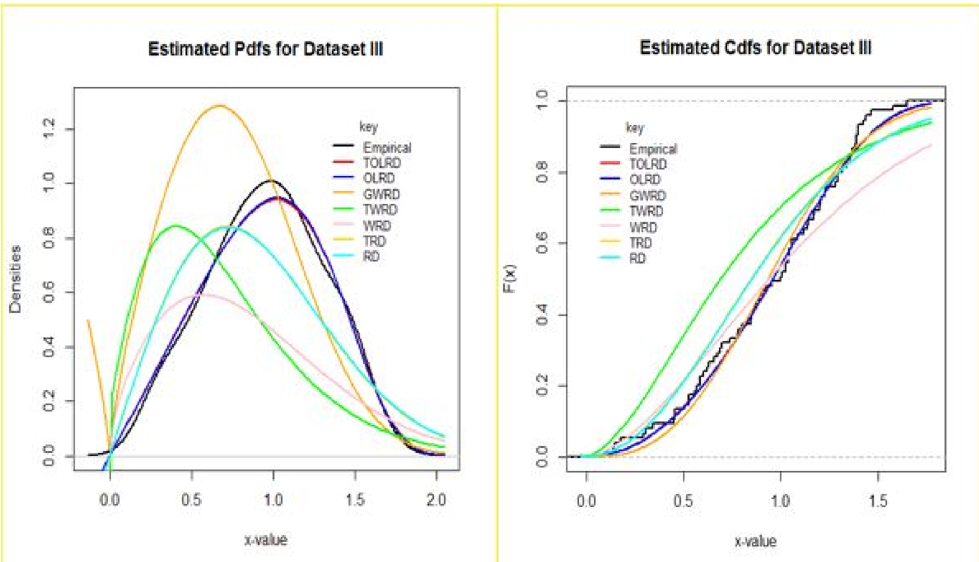

Figure 7: : Plots of the estimated densities and cdfs of the TOLRD and other fitted distributions to dataset III

transmuted odd Lindley-Rayleigh distribution (TOLRD) performs much better for the real life dataset (dataset I) compared to the other six fitted distributions (odd Lindley-Rayleigh distribution (OLRD), generalized Weibull-Rayleigh distribution (GWRD), transmuted Weibull-Rayleigh distribution (TWRD), Weibull-Rayleigh distribution (WRD), transmuted Rayleigh distribution (TRD) and the conventional Rayleigh distribution (RD)), meanwhile there is no much difference between the performance of the transmuted odd Lindley-Rayleigh distribution (TOLRD) and that of the odd Lindley-Rayleigh distribution (OLRD) for the two simulated datasets (datasets II and III) despite the slight variation in the two families though the TOLRD and OLRD are still much better in performance compared to the other fitted six distributions under the

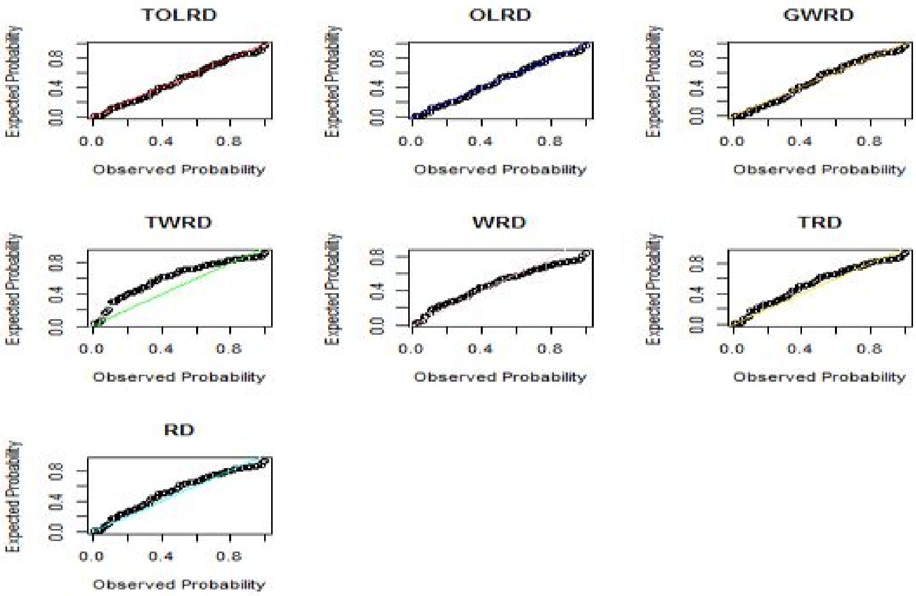

Figure 8: Probability plots for the fit of the TOLRD and other fitted models based on dataset III.

two simulated datasets. Based on these results, it is proven that the proposed distribution (transmuted odd Lindley-Rayleigh distribution (TOLRD)) is a more flexible distribution than the other existing distributions also fitted in this study. Also these results are clearly confirmed by the estimated density plots and also the probability plots of the fitted distributions as shown in the figures above.

4. Conclusion

In this paper, a new distribution has been proposed named (Transmuted odd Lindley-Rayleigh distribution) . Some mathematical and statistical properties of the proposed distribution have been studied appropriately. The derivations of some expressions for its moments, moment generating function, characteristics function, survival function, hazard function, quantile function and ordered statistics has been done appropriately. Some plots of the distribution revealed that it can take various shapes depending on values of the parameters. The model parameters have been estimated using the method of maximum likelihood estimation. The implications of the plots for the survival function indicate that the transmuted odd Lindley-Rayleigh distribution has a decreasing survival rate and an increasing failure function. The results of the applications to two simulated datasets showed that the proposed distribution (Transmuted odd Lindley-Rayleigh distribution) fits the two datasets better compared to the other six fitted distributions (odd Lindley-Rayleigh distribution (OLRD), generalized Weibull-Rayleigh distribution (GWRD), transmuted Weibull-Rayleigh distribution (TWRD), Weibull-Rayleigh distribution (WRD), transmuted Rayleigh distribution (TRD) and the conventional Rayleigh distribution (RD)) irrespective of the nature of the data sets and their sample sizes. This indicates that the TOLRD could be useful for modeling different types of data.

References

Abdel-Hady, D. H. (2013). Bivariate generalized Rayleigh distribution. Journal of Applied Sciences Research, **9**(9): 5403-5411.

Afify, A. Z. and Aryal, G. (2016). The Kummaraswamy exponentiated Frechet distribution. Journal of Data Science, **6**: 1-19

Afify, M. Z., Yousof, H. M., Cordeiro, G. M., et al. (2016). The Weibull Frechet distribution and its applications. Journal of Applied Statistics.

Ahmad, A., Ahmad, S. P. and Ahmed, A. (2014). Transmuted inverse Rayleigh distribution: a generalization of the inverse Rayleigh distribution. Mathematical Theory and Modeling, **4**(7): 90-98.

Alizadeh, M., Cordeiro, G. M., Pinho, L. G. B. and Ghosh, I. (2017). The Gompertz-G family of distributions. Journal of Statistical Theory and Practice, **11**(1): 179–207.

Alzaatreh, A., Famoye, F. and Lee, C. (2013). Weibull-Pareto distribution and its applications. Communications in Statistics: Theory and Methods, **42**: 1673–1691

Alzaatreh, A., Famoye, F. and Lee, C. (2014). The gamma-normal distribution: properties and applications. Computational Statistics and Data Analysis, **69**: 67–80.

Ashour, S. K. and Eltehiwy, M. A. (2013). Transmuted Lomax distribution. American Journal of Applied Mathematics and Statistics, **1**(6): 121-127.

Barreto-Souza, W. M., Cordeiro, G. M. and Simas, A. B. (2011). Some results for beta Frechet distribution. Communication in Statistics: Theory and Methods, **40**: 798-811

Bourguignon, M., Silva, R. B. and Cordeiro, G. M. (2014). The Weibull-G family of probability distributions. Journal of Data Science, **12**: 53-68.

Cakmakyapan, S. and Ozel, G. (2016). The Lindley family of distributions: properties and applications. Hacettepe Journal of Mathematics and Statistics, **46**: 1-27.

Cordeiro, G. M., Afify, A. Z., Ortega, E. M. M., Suzuki, A. K. and Mead, M. E. (2019). The odd Lomax generator of distributions: properties, estimation and applications. Journal of Computational and Applied Mathematics, **347**: 222–237.

Cordeiro, G. M., Ortega, E. M. M., Popovic, B. V. and Pescim, R. R. (2014). The Lomax generator of distributions: properties, minification process and regression model. Applied Mathematics and Computation, **247**: 465-486.

Corless, R. M., Gonnet, G. H., Hare, D.E.G., Jeffrey, D.J. and Knuth, D.E. (1996). On the Lambert W function. Advances in Computational Mathematics, **5**: 329-359.

Gomes-Silva, F., Percontini, A., De Brito, E., Ramos, M. W., Venancio, R. and Cordeiro, G. M. (2017). The odd Lindley-G family of distributions. Austrian Journal of Statistics, **46**: 65-87.

Hirano, K. (1986). Rayleigh distributions. John Wiley, New York.

Howlader, H. A. and Hossain, A. (1995). On Bayesian estimation and prediction from Rayleigh distribution based on type-II censored data. Communication Statistics: Theory and Methods, **24**(9): 2249-2259.

Hyndman. R.J. and Fan, Y. (1996). Sample quantiles in statistical packages, The American Statistician, **50**(4): 361-365.

Ieren, T. G. and Abdullahi, J. (2020a). Properties and applications of a two-parameter inverse exponential distribution with a decreasing failure rate. Pakistan Journal of Statistics, **36**(3): 183-206.

Ieren, T. G. and Abdullahi, J. (2020b). A transmuted normal distribution: properties and applications. Equity Journal of Science and Technology, **7**(1): 16-35.

Ieren, T.G., Abdulkadir, S. S and Issa, A. A. (2020). Odd Lindley-Rayleigh distribution: its properties and applications to simulated and real life datasets. Journal of Advances in Mathematics and Computer Science, **35**(1): 63-88.

Kenney, J. F. and Keeping, E. S. (1962). Mathematics of statistics (3 ed.), Chapman & Hall, New Jersey.

Koleoso, P. O., Chukwu, A. U. and Bamiduro, T. A. (2019). A three-parameter Gompertz-Lindley distribution: its properties and applications. Journal of Mathematical Theory and Modeling, **9**(4): 29-42.

Kundu, D. and Raqab, M. Z. (2005). Generalized Rayleigh distribution: different methods of estimations. Computational Statistics and Data Analysis, **49**: 187 – 200.

Merovci, F. and Elbatal, I. (2015). Weibull Rayleigh distribution: theory and applications. Applied Mathematics and Information Science, **9**(5): 1-11.

Merovci, F. (2013). The transmuted rayleigh distribution. Australian Journal of Statistics, **22**(1): 21–30

Moors, J. J. (1988). A quantile alternative for kurtosis. Journal of the Royal Statistical Society: Series D, **37**: 25–32.

Nichols, M. D. and Padgett, W. J. (2006). A bootstrap control chart for Weibull percentiles. Quality Reliability Engineering Int., **22**: 141-151.

Oguntunde, P. E., Balogun, O. S., Okagbue, H. I., and Bishop, S. A. (2015). The Weibull-Exponential distribution: its properties and applications. Journal of Applied Sciences, **15**(11): 1305-1311.

Rayleigh, J. (1980). On the resultant of a large number of vibrations of the same pitch and of arbitrary phase. Philosophy and Management, **10**: 73–78.

Shaw, W. T. and Buckley, I. R. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map. Research report.

Siddiqui, M. M. (1962). Some problems connected with Rayleigh distributions. Journal of Research National Bureau of the Standards, **60**D: 167–174.

Smith, R. L. and Naylor, J. C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. Journal of Applied Statistics, **36**: 358-369.

Tahir, M. H., Zubair, M., Mansoor, M., Cordeiro, G. M. and Alizadeh, M. (2016). A new Weibull-G family of distributions. Hacettepe Journal of Mathematics and Statistics, **45**(2): 629-647.

Yahaya, A. and Alaku, A. Y. (2018). On generalized Weibull-Rayleigh distribution: its properties and applications. ATBU Journal of Science, Technology and Education, **5**(1): 217-231.

Yahaya, A. and Ieren, T. G. (2017). On transmuted Weibull-Exponential distribution: its properties and applications, Nigerian Journal of Scientific Research, **16**(3): 289-297.

Yahaya, A., Abdullahi, J. and Ieren, T. G. (2020). Properties and applications of a transmuted Weibull–Rayleigh distribution. Science Forum (Journal of Pure and Applied Sciences), **19**: 126–138.