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On the Lomax-Kumaraswamy distribution

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Abstract. This article presents a four-parameter distribution called the Lomax-Kumaraswamy distribution. Some statistical properties of the proposed distribution have been studied. The derivations of some expressions for its moments, moment generating function, characteristics function, survival function, hazard function, quantile function and ordered statistics has been done alongside with the theoretical and graphical illustrations. Some plots of the distribution revealed that it is a positively skewed distribution. This study also estimated the parameters of the new distribution using the method of maximum likelihood. Finally, the study evaluated the performance of the new model and compared it to the baseline and some extensions of the Kumaraswamy distribution using two real life datasets.

Keywords: Lomax-Kumaraswamy distribution, moments, reliability analysis, order statistics, maximum likelihood estimation.

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1. Background

The Kumaraswamy probability distribuion has the same basic properties as the beta distribution. It was first proposed by a researcher named Poondi Kumaraswamy in 1980, as an alternative to beta distribution (Khan et al.,2016). Kumaraswamy distribution is a type in which outcomes are limited to a specific range and the probability density function within this range being characterized by two shape parameters. It is similar to the beta distribution but easier to use because it has simpler analytical expressions for both its probability density function and cumulative distribution function. The distribution is the most widely applied in hydrological problems and many natural phenomena whose process values are bounded on both sides. The Kumaraswamy double bounded distribution denoted by Kw(x; a, b) is a family of continuous probability distributions defined on the interval (0,1) with cumulative distribution function (cdf) and probability density function (pdf) (Kumaraswamy, 1980) are:

$$G(x) = 1 - (1 - x^a)^b; 0 < x < 1, a, b > 0 \tag{1}$$

and

$$g(x) = abx^{a-1}(1 - x^a)^{b-1}; 0 < x < 1, a, b > 0 \tag{2}$$

respectively, where a and b are the shape parameters.

There are many generalized families of distributions with better performance that were proposed by different researchers. Some of them include: Kumaraswamy Generalized (Kw-G) family distributions (Cordeiro and De Castro 2009), the generalized Weibull (Weibull-G) family of distributions

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(Bourguignon et al., 2014), the Kumaraswamy-G Poisson (Kw-GP) (Ramos et al.,2015), and a Lomax generator (Lomax-G) (Cordeiro et al., 2014), to generalize any continuous baseline distribution.

There are also some extensions of the Kumaraswamy distribution in the literature such as the Kumaraswamy-Kumaraswamy distribution (El-Sherpieny and Ahmad, 2014), transmuted Kumaraswamy distribution (Khan et al., 2016) and the exponentiated-Kumaraswamy distribution (Javanshiri et al., 2015).

This article is aimed at studying the properties of a new extension of the Kumaraswamy distribution introduced by Cordeiro et al. (2014). The distribution is commonly called Lomax-Kumaraswamy distribution (LKwD). The remaining parts of this paper are presented in sections as follows: We defined the new distribution and give its plots in section 2. Section 3 derived some properties of the new distribution. The estimation of parameters using maximum likelihood estimation (MLE) is provided in section 4. In section 5, we carried out application of the proposed distribution with two real life data sets. Lastly, in section 6, we give some concluding remarks.

The rationale for studying the Lomax Kumaraswamy distribution (LKwD), is that we have added a new extention to the Kumaraswamy family of distributions. The new Lomax-Kumaraswamy distribution has two additional parameters which helps in capturing the main features of the data such as the skewness, and also increased the flexibility of the baseline distribution to model more easily and appropriately data sets that do not properly fit the kumaraswamy distribution, which is an improvement on the baseline distribution.

2. The Lomax-Kumaraswamy distribution (LKwD)

Here, we derived the cdf and pdf of the Lomax-Kumaraswamy distribution using the Lomax generator proposed by Cordeiro et al. (2014). According to them the cdf and pdf of the Lomax-G family (Lomax-based generator) for any continuous probability distribution are given by:

$$F(x) = \int_0^{-log[1-G(x)]} \alpha\beta^{\alpha} \frac{dt}{(\beta+t)^{\alpha+1}} = 1 - \left(\frac{\beta}{\beta - log[1-G(x)]}\right)^{\alpha} \tag{3}$$

and

$$F(x) = \alpha\beta^{\alpha} \frac{g(x)}{[1-G(x)]\,(\beta - log[1-G(x)])^{\alpha+1}} \tag{4}$$

respectively, where g(x) and G(x) are the pdf and cdf of any continuous distribution to be generalized respectively and and are the two additional new parameters responsible for the scale and shape of the distribution respectively. Substituting equations (1) and (2) in (3) and (4) and simplifying, we obtain the cdf and pdf of the Lomax-Kumaraswamy distribution as follows:

$$F(x) = 1 - \frac{\beta^{\alpha}}{(\beta - log(1-x^a)^b)^{\alpha}}; 0 < x < 1 \tag{5}$$

and

$$f(x) = \alpha\beta^{\alpha} abx^{a-1}(1-x^a)^{-1}\left[\beta - log(1-x^a)^b\right]^{-(\alpha+1)}; 0 < x < 1 \tag{6}$$

respectively, for , where are the shape parameters, while is the scale parameter. Given some values for the parameters and , we provide some possible shapes for the pdf and the cdf of the LKwD as shown in figure 1 and 2 below:

Figure 1 indicates that the LKwD distribution can assume various shapes such as asymmetrical, left-skewed, right-skewed shapes. This means that distribution can be very useful for datasets with different shapes.

From the above cdf plot, the cdf approaches one as the value of X increases. This implies that the cdf is monotonically non-decreasing as expected from any probability distribution.

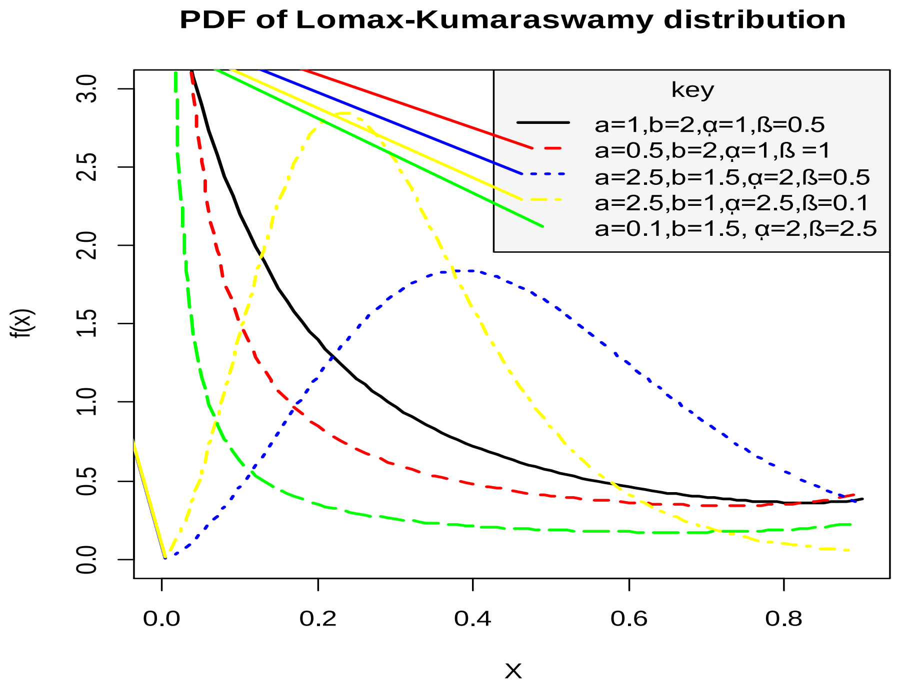

Figure 1.: pdf of the LKwD for different values of the parameters

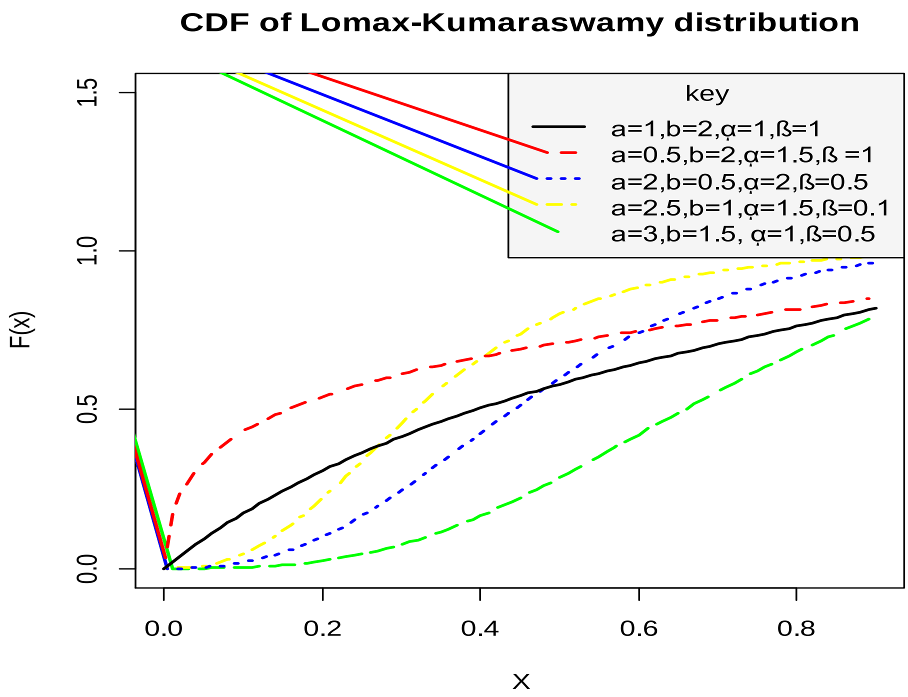

Figure 2.: cdf of the LKwD for different values of the parameters

3. Properties

In this section, we discuss some properties of the LKwD distribution.

3.1 *Moments*

Let X denote a continuous random variable, the nth moment of X is given by;

$$\mu_n = E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx = \int_{0}^{\infty} x^n f(x)dx \tag{7}$$

Recall that,

$$f(x) = \alpha\beta^{\alpha} abx^{a-1}(1-x^a)^{-1}\beta^{-(\alpha+1)}\left[1-\beta^{-1}log(1-x^a)^b\right]^{-(\alpha+1)} \tag{8}$$

Hence, let

$$A = \left[1-\beta^{-1}log(1-x^a)^b\right]^{-(\alpha+1)} \tag{9}$$

Using the generalized binomial theorem above we have:

$$A = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+i+1)}{j!\Gamma(\alpha+1)}\beta^{-i}\left(log\left[1-x^a\right]^b\right)^i \tag{10}$$

Now, consider the following formula which holds for $i \geq 1$, we can write the last term in (10) above as

$$\left(log(1-x^a)^b\right)^i = \sum_{k,l=0}^{\infty}\sum_{j=0}^{\infty} \frac{i}{i-j}\binom{k-i}{k}\binom{k}{j}\binom{k+i}{l}P_{j,k}\left[(1-x^a)^b\right]^l \tag{11}$$

Where for $j \geq 0$, $P_{j,0} = 1$ and for $k = 1, 2, ..$

$$P_{j,k} = k^{-1}\sum_{m=1}^{k}(-1)^m\left(\frac{m(j+1)-k}{m+1}\right)P_{j,k-m} \tag{12}$$

Combining equation (10) and (11) and inserting the above power series in equation (8), we have:

$$f(x) = \alpha\beta^{-1}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\alpha+i+1)\beta^{-i}}{j!\Gamma(\alpha+1)}\frac{i}{i-j}\binom{k-i}{k}\binom{k}{j}\binom{k+i}{l}P_{j,k}abx^{a-1}(1-x^a)^{bl-1} \tag{13}$$

$$= \sum_{l=0}^{\infty} W_{i,j,k}abx^{a-1}(1-x^a)^{bl-1} \tag{14}$$

where

$$W_{i,j,k} = \alpha\beta^{-1}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\alpha+i+1)\beta^{-i}}{j!\Gamma(\alpha+1)}\frac{i}{i-j}\binom{k-i}{k}\binom{k}{j}\binom{k+i}{l}P_{j,k} \tag{15}$$

Hence

$$\mu_n = E[X^n] = \int_0^1 x^n f(x)dx = \sum_{l=0}^{\infty} W_{i,j,k}ab\int_0^1 x^{n+a-1}(1-x^a)^{bl-1}dx \tag{16}$$

Recall that for the Kumaraswamy distribution (Khan et al., 2016);

$$E[X^r] = \int_0^1 x^r f(x)dx = ab\int_0^1 x^{r+a-1}(1-x^a)^{b-1}dx = b\beta\left(\frac{r}{a}+1, b\right) \tag{17}$$

Hence, this implies that

$$\mu_n = E[X^n] = \int_0^1 x^n f(x)dx = \sum_{l=0}^{\infty} W_{i,j,k} ab \int_0^1 x^{n+a-1}(1-x^a)^{bl-1}dx \tag{18}$$

$$= \sum_{l=0}^{\infty} W_{i,j,k} ab \int_0^1 x^{n+a-1}(1-x^a)^{bl-1}dx \tag{19}$$

$$= \sum_{l=0}^{\infty} W_{i,j,k} b\beta\left(\frac{n}{a}+1, bl\right) \tag{20}$$

3.1.1 The mean

The mean of the LKwD can be obtained from the nth moment of the distribution when $n = 1$ as follows:

$$\mu_1 = E[X^1] = \sum_{l=0}^{\infty} W_{i,j,k} b\beta\left(\frac{1}{a}+1, bl\right) \tag{21}$$

Also the second moment of the LKwD is obtained from the nth moment of the distribution when $n = 2$ as

$$E[X^2] = \sum_{l=0}^{\infty} W_{i,j,k} b\beta\left(\frac{2}{a}+1, bl\right) \tag{22}$$

3.1.2 The variance

The nth central moment or moment about the mean of X, say μ_n, can be obtained as

$$\mu_n = E[X-\mu_1]^n = \sum_{i=0}^{n}(-1)^i \tbinom{n}{i} \mu_1'^{i} \mu_{n-i}' \tag{23}$$

The variance of X for LKwD is obtained from the central moment when $n = 2$, that is,

$$Var(X) = E(X^2) - (E(X))^2 \tag{24}$$

$$Var(X) = \sum_{l=0}^{\infty} W_{i,j,k} b\beta\left(\frac{2}{a}+1, bl\right) - \left(\sum_{l=0}^{\infty} W_{i,j,k} b\beta\left(\frac{1}{a}+1, bl\right)\right)^2 \tag{25}$$

The variation, skewness and kurtosis measures can also be calculated from the non-central moments using some well-known relationships.

3.2 *Moment generating function*

The mgf of a random variable X can be obtained as

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)dx \tag{26}$$

where

$$e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n \tag{27}$$

Using the power series expansion of equation (27) and simplifying the integral in (26), we have;

$$M_x(t) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^n}{n!} W_{i,j,k} b\beta \left(\frac{n}{a} + 1, bl\right) \tag{28}$$

Where n and t are constants, t is a real number and denotes the nth ordinary moment of X.

3.3 *Characteristics function*

The characteristics function of a random variable X is given by;

$$\phi_x(t) = E(e^{itx}) = E[cos(tx) + isin(tx)] = E[cos(tx)] + iE[sin(tx)] \tag{29}$$

Using power series expansion

$$\phi_x(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mu_{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \mu_{2n+1} \tag{30}$$

where μ'_{2n} and μ'_{2n+1} are the moments of X for $n = 2n$ and $n = 2n + 1$ respectively.

$$\phi_x(t) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^{2n}}{(2n)!} W_{i,j,k} b\beta \left(\frac{2n}{a} + 1, bl\right) + \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} W_{i,j,k} b\beta \left(\frac{2n+1}{a} + 1, bl\right) \tag{31}$$

3.4 *Quantile function for the LKwD*

The quantile function, say $X = Q(u)$, of the LKwD can be obtained as the inverse of the cdf

$$F(x) = 1 - \frac{\beta^{\alpha}}{[\beta - log\,(1 - (1 - (1 - x^a)^b))]^{\alpha}}$$

as;

$$Q(u) = X_q = \sqrt[a]{\left(1 - \left[\exp\left\{\beta - \frac{\beta}{(1-u)^{\frac{1}{\alpha}}}\right\}\right]^{\frac{1}{b}}\right)} \tag{32}$$

Using (31) above, the median of X from the LKwD is simply obtained by setting u=0.5 while random numbers can be generated from LKwD by setting , where u is a uniform variate on the unit interval (0,1).

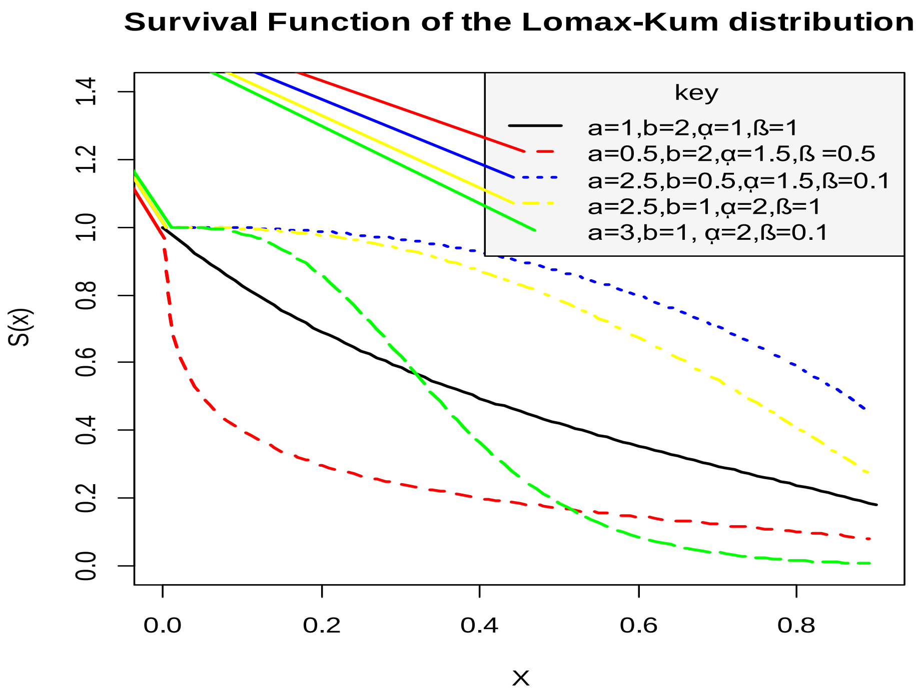

Figure 3.: The survival function of the LKwD for different values of the parameters

3.5 *Skewness and kurtosis*

The quantile based measures of skewness and kurtosis will be employed. This is due to the non-existence of the classical measures in some cases. The Bowley's measure of skewness (Kennedy and Keeping, 1962.) based on quartiles is given by;

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{33}$$

and the Moores (1998) kurtosis is on octiles and is given by;

$$KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{4}\right)} \tag{34}$$

3.6 *Reliability analysis*

This subsection presents the survival and hazard functions for the LKwD.
The survival function: is the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = P(X > x) = 1 - F(x) \tag{35}$$

Where $F(x)$ is the cdf of the distribution. The survival function for the LKwD is obtained as:

$$S(x) = \frac{\beta^{\alpha}}{\left[\beta - log\left(1 - \left(1 - \left(1 - x^{a}\right)^{b}\right)\right)\right]^{\alpha}} \tag{36}$$

Below is a plot of the survival function at chosen parameter values in Figure 3 The figure above revealed that the probability of survival for any random variable following a Lomax-Kumaraswamy distribution decreases as the values of the random variable X increases. This implies that the Lomax-

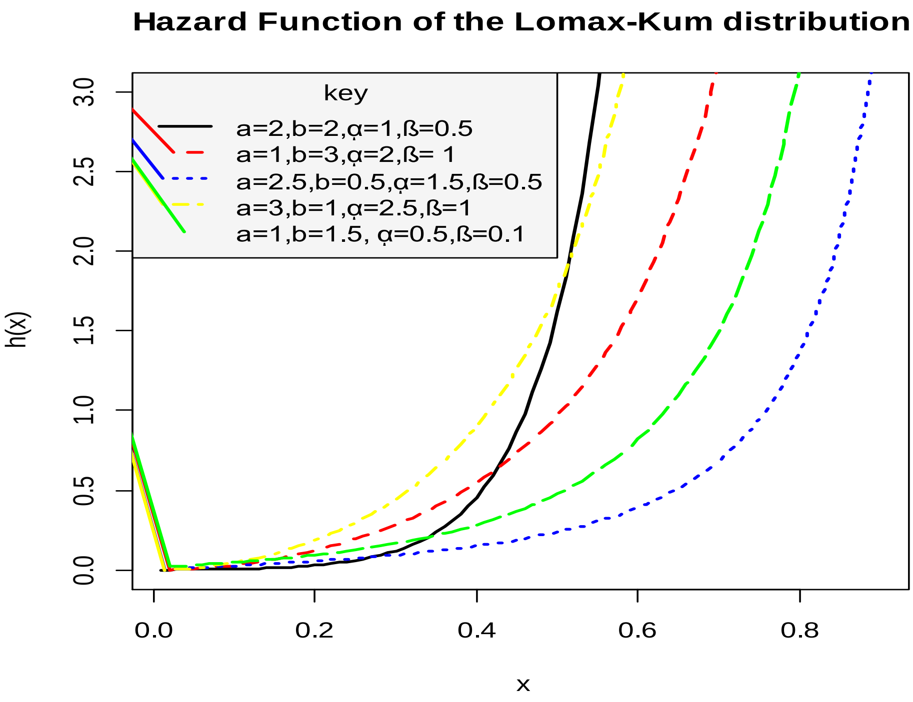

Figure 4.: The hazard function of the LKwD for different values of the parameters

Kumaraswamy distribution can be used to model random variables whose survival rate decreases as their age grows.

3.7 *The hazard function*

The hazard function: is the rate of failure of a component defined over the entire domain of the random variable X. The hazard function is defined as:

$$\frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{37}$$

Substituting for f(x) and F(x) and simplifying we obtained

$$h(x) = \frac{ab\alpha x^{a-1}\left(\beta - log\left[1 - x^a\right]^b\right)^{-1}}{1 - x^a} \tag{38}$$

3.8 *Order statistics*

In this section, we derive closed form expressions for the pdf of the ath order statistics of the LKwD. If $X_1, X_2, .., X_n$ is a random sample from the LKwD and also let $X_{1:n}, X_{2:n}, .., X_{i:n}$ be the corresponding order statistic obtained from this sample. The pdf, $f_{i:n}$ of the ath order statistic can be obtained as:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\left[F(x)\right]^{k+i-1}f(x) \tag{39}$$

where $F(x)$ and $f(x)$ are the cdf and pdf as defined in (5) and (6) respectively. Using (5) and (6), the pdf of the ith order statistics $X_{i:n}$ can be expressed from (39) as;

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \frac{n!}{(i-1)!(n-i)!}(-1)^k \binom{n-i}{k} \left[1 - \frac{\beta^\alpha}{[\beta - log\,(1-(1-(1-x^a)^b))]^\alpha}\right]^{k+i-1} \times$$

$$\left[\sum_{l=0}^{\infty} W_{i,j,k} abx^{a-1}(1-x^a)^{b-1} \left[log(1-x^a)^b\right]^{l-1}\right] \tag{40}$$

Hence, the pdf of the minimum order statistic $X_{1:n}$ and maximum order statistic $X_{n:n}$ of the LKwD are given by;

$$f_{1:n}(x) = n \sum_{k=0}^{n-i} (-1)^k \binom{n-1}{k} \left[1 - \frac{\beta^\alpha}{[\beta - log\,(1-(1-(1-x^a)^b))]^\alpha}\right]^{k} \times$$

$$\left[\sum_{l=0}^{\infty} W_{i,j,k} abx^{a-1}(1-x^a)^{b-1} \left[log(1-x^a)^b\right]^{l-1}\right] \tag{41}$$

and

$$f_{n:n}(x) = n \left[1 - \frac{\beta^\alpha}{[\beta - log\,(1-(1-(1-x^a)^b))]^\alpha}\right]^{n-1} \left[\sum_{l=0}^{\infty} W_{i,j,k} abx^{a-1}(1-x^a)^{b-1} \left[log(1-x^a)^b\right]^{l-1}\right] \tag{42}$$

4. Estimation of parameters

Let $X_i = x_1, x_2, ...x_n$ be a sample of size 'n' of the independently and identically distributed random variables from the LKwD with unknown parameters α, β, a, and b defined previously the pdf of the LKwD is given as (6) while its likelihood function $L(x_1, x_2, ...x_n/\alpha, \beta, a, b)$ is given by

$$L(x_i/\alpha, \beta, a, b) = (\alpha\beta^\alpha ab)^n \prod_{i=1}^{n} x_i^{a-1}(1-x_i^a)^{-1} \left(\beta - log\,(1-x_i^a)^b\right)^{-(\alpha+1)} \tag{43}$$

$$L(x_i/\alpha, \beta, a, b) = (\alpha\beta^\alpha ab)^n \prod_{i=1}^{n} x_i^{a-1} \prod_{i=1}^{n}(1-x_i^a)^{-1} \prod_{i=1}^{n} \left(\beta - log\,(1-x_i^a)^b\right)^{-(\alpha+1)} \tag{44}$$

Let the log-likelihood function be $l = LogL(x_i/\alpha, \beta, a, b)$, then taking the natural logarithm of the likelihood function, we obtained

$$l = nlog\alpha + n\alpha log\beta + nloga + nlogb + (a-1)\sum_{i=1}^{n} logx_i - \sum_{i=1}^{n} log(1-x_i^a) - (\alpha+1)\sum_{i=1}^{n} log\left(\beta - log(1-x_i^a)^b\right) \tag{45}$$

Differentiating l partially with respect to α, β, a and b respectively gives;

$$\frac{\delta l}{\delta a} = \frac{n}{a} + \sum_{i=1}^{n} logx_i + \sum_{i=1}^{n} \frac{x_i^a logx_i}{(1-x_i^a)} + b(\alpha+1)\sum_{i=0}^{n} \left(\frac{x_i^a logx_i}{(\beta - log(1-x_i^a)^b)\,(1-x_i^a)}\right) \tag{46}$$

$$\frac{\delta l}{\delta b} = \frac{n}{b} + (\alpha+1)\sum_{i=0}^{n} \left(\frac{log(1-x_i^a)}{(\beta - log(1-x_i^a)^b)}\right) \tag{47}$$

Table 1.: Summary of the two data sets

Parameters	n	Min	Q_1	Median	Q_3	Mean	Max	Variance	Skewness	Kurtosis
Data set I	20	0.265	0.335	0.407	0.458	0.423	0.74	0.0157	1.068	0.6
Data set II	48	0.09	0.162	0.199	0.263	0.218	0.464	0.007	1.169	1.11

$$\frac{\delta l}{\delta\alpha} = \frac{n}{\alpha} + nlog\beta - \sum_{i=1}^{n} log\left(\beta - log(1 - x_i^a)^b\right) \tag{48}$$

$$\frac{\delta l}{\delta\beta} = \frac{n\alpha}{\beta} - (\alpha + 1)\sum_{i=0}^{n} \frac{1}{(\beta - log(1 - x_i^a)^b)} \tag{49}$$

The solution of the non-linear system of equations in (46), (47), (48) and (49) will give the maximum likelihood estimates of parameters and However, the solution cannot be obtained analytically except numerically with the aid of suitable statistical software. In this paper we used R-package.

5. Applications

This section presents the application of the Lomax-Kumaraswamy distribution (LKwD) using two datasets and compares the performance of the LKwD to the transmuted Kumaraswamy distribution (TKwD), the Kumaraswamy-Kumaraswamy distribution (KwKwD) and the Kumaraswamy distribution (KwD). The two datasets are obtained from the literature as follows:

Data set I:

This dataset comprises of flood data with 20 observations measured in cubic meters per second (m3/s) obtained from Dumonceaux and Antle (1973). It was used by Khan et al. (2016). The data are as follows: 0.265 0.269 0.297 0.315 0.324 0.338 0.379 0.379 0.392 0.402 0.412 0.416 0.418 0.423 0.449 0.484 0.494 0.613 0.654 0.740.

Data set II:

The second data set is on shape measurements of 48 rock samples in meters (m) from a petroleum reservoir. This data was extracted from BP research, image analysis by Ronit Katz (R-library/datasets/htm/rock.html) and it was used by Javanshiri et al. (2015). as follows: 0.0903296 0.189651 0.228595 0.200071 0.280887 0.311646 0.176969 0.464125 0.148622 0.164127 0.231623 0.144810 0.179455 0.276016 0.438712 0.420477 0.183312 0.203654 0.172567 0.113852 0.191802 0.19753 0.163586 0.200744 0.117063 0.162394 0.153481 0.291029 0.133083 0.326635 0.253832 0.262651 0.122417 0.150944 0.204314 0.240077 0.225214 0.154192 0.328641 0.182453 0.167045 0.148141 0.262727 0.161865 0.341273 0.276016 0.230081 0.200447.

The summary of the two data sets is provided in Table 1. We also provide some histograms and frequency curves for the two data sets as shown in Figure 5 and 6 below respectively.

From the descriptive statistics, histograms and densities shown above for the two data sets, we observed that both the first and second data sets are positively skewed and therefore suitable for distributions that are skewed to the right. For us to assess the distributions listed above, we made use of some model selection criteria: the AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan Quin information criterion). The estimates and performance measures of the above distributions are listed in Tables 2 and 3 for datasets I and II follows

From Figures 7 and 8 above, we can clearly see that the performance of the Lomax-Kumaraswamy distribution (LKwD) and the Kumaraswamy-Kumaraswamy distribution (KwKwD), remained con-

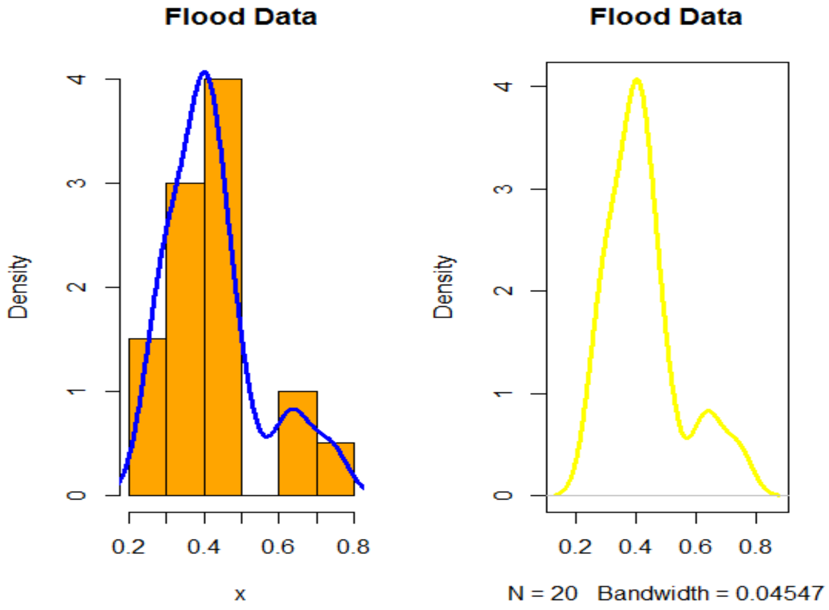

Figure 5.: A histogram and density plot for the flood data (Data set I)

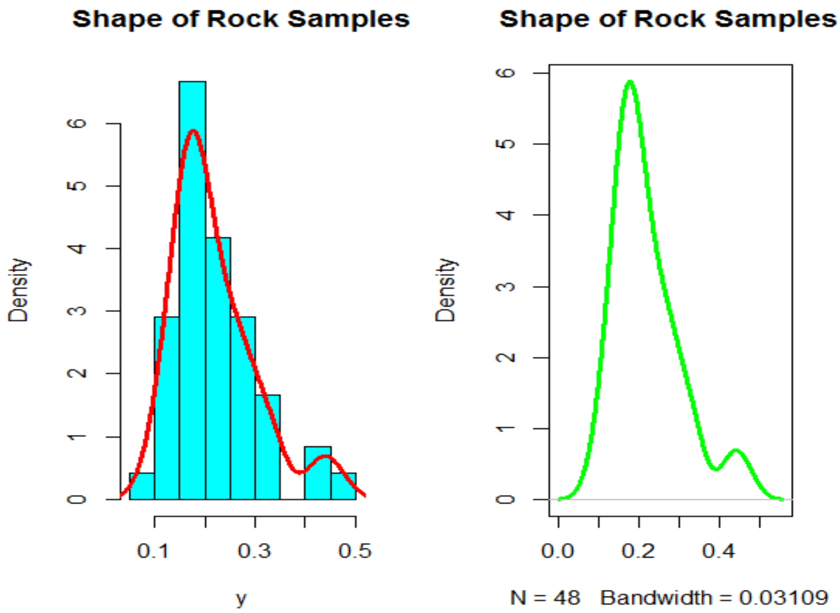

Figure 6.: A histogram and density plot for the rock sample data (Data set II)

sistent as the competing distributions that fits the two data sets, followed by the Transmuted Kumaraswamy (TKwD), distribution while the baseline Kumaraswamy distribution (KwD) came last. Having demonstrated earlier in Tables 2 and 3 with the values of their AIC, CAIC, BIC and HQIC, we have a similar conclusion on Figure 7 and 8. The Lomax-Kumaraswamy distribution performed competitively with Kumaraswamy-Kumaraswamy distribution, and outperformed the Transmuted Kumaraswamy and the Kumaraswamy distributions for the two data sets considered in this study.

6. Conclusion

In this study, we introduced a new extension of the Kumaraswamy distribution with two additional parameters using the Lomax generator proposed by Cordeiro et al. (2014). This study has

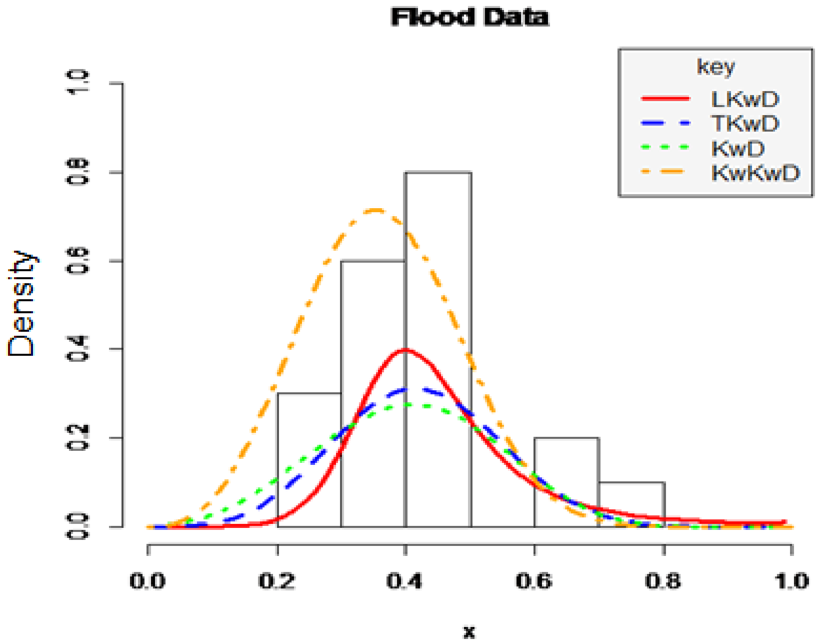

Figure 7.: Histogram and estimated densities of the LKwD, TKwD, KwKwD and KwD for 20 observations of the flood data (dataset I)

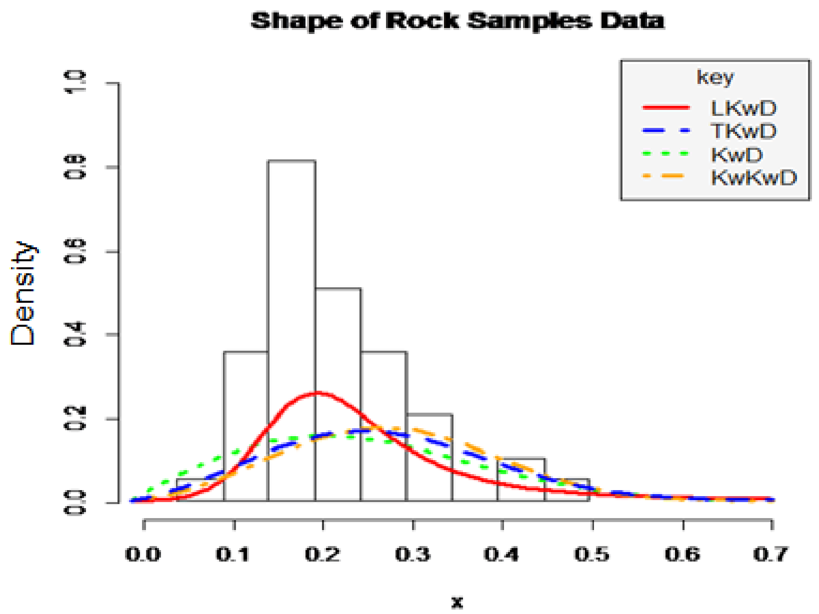

Figure 8.: Histogram and estimated densities plots of the LKwD, TKwD,KwKwDand KwD for the shape of rock samples (data set II)

derived some expressions for its basic statistical properties such as moments, moment generating function, the characteristics function, reliability analysis, quantile function and the distribution of order statistics. Some plots of the distribution revealed that it is a positively skewed distribution. The model parameters have been estimated using the method of maximum likelihood estimation. The implications of the plots for the survival function indicate that the Lomax-Kumaraswamy distribution could be used to model time or age-dependent events, where survival rate decreases with time or age. The performance of the Lomax-Kumaraswamy distribution was tested by using to two real data sets in the literature. The results showed that it can serve as an alternative distribution to model positively skewed data sets.

Table 2. MLEs and Performance of the distributions based on the data set I.

Distributions	Parameter estimates	-ll(log-likelihood value)	AIC	BIC	CAIC	HQIC	Ranks of models performance
LKwD	$\hat{a}$=7.3061 $\hat{b}$=6.2384 $\hat{\alpha}$=0.7269 $\hat{\beta}$=0.0779	-15.9278	-23.8556	-26.6515	-21.1889	-30.9413	2
TKwD	$\hat{a}$=2.1658 $\hat{b}$=8.9877 $\hat{\lambda}$=0.4875	-14.1321	-22.2642	-24.3611	-21.142	-27.57	3
KwD	$\hat{a}$=2.7925 $\hat{b}$=10.2153	-12.8416	-21.6832	-23.0811	-20.9773	-25.2261	4
KwKwD	$\hat{a}$=2.0168 $\hat{b}$=6.1956 $\hat{\alpha}$=3.3247 $\hat{\beta}$=1.8551	-16.3247	-24.6494	-27.4453	-21.9827	-31.7351	1

Table 2, shows that the best performing model is the KwKwD, with the lowest values of AIC, CAIC, BIC and HQIC ,while LKwDhas better performance compared to the TKwD and KwD.

Table 3. MLEs and Performance of the distributions based on the data set II.

Distributions	Parameter estimates	-ll=(log-likelihood value)	AIC	BIC	CAIC	HQIC	Ranks based on models performance
LKwD	$\hat{a}$=4.6170 $\hat{b}$=9.1743 $\hat{\alpha}$=1.0974 $\hat{\beta}$=0.0081	-57.3044	-106.6088	-107.8838	-105.6786	-112.8038	1
TKwD	$\hat{a}$=1.8535 $\hat{b}$=2.3715 $\hat{\lambda}$=0.0799	-50.2138	-94.4276	-95.3839-	-93.882	-99.0739	3
KwD	$\hat{a}$=1.8978 $\hat{b}$=12.5398	-47.3652	-90.7304	-91.3679	-90.4637	-93.8279	4
KwKwD	$\hat{a}$=9.2497 $\hat{b}$=1.9360 $\hat{\alpha}$=0.2917 $\hat{\beta}$=15.9876	-52.4279	-96.8558	-98.1308	-95.9256	-103.0508	2

Table 3 indicates that the LKwD has better performance with the lowest values of AIC, CAIC, BIC and HQIC compared TKwD, KwKwD and KwD.

REFERENCES

Bourguignon, M.,Silva. R. B. and Cordeiro, G. M. (2013). The Kumaraswamy Pareto distribution, Journal of Statistical Theory and Applications, **12**(3): 129-144.

Cordeiro, G. M., De Castro, M. (2009). A new family of generalized distributions, Journal of Statistical Computation and Simulation: 1–17.

Cordeiro, G. M., Ortega, E. M. M., Popovic, B. V. and Pescim, R. R. (2014). The Lomax generator of distributions: Properties, minification process and regression model. Applied Mathematics and Computation, **247**: 465-486.

Dumonceaux, R. and Antle, C. E. (1973). Discrimination between the Log-Normal and the Weibull Distributions. Technometrics, **15**(4): 923-926.

El-Sherpieny, E. E. and Ahmed, M. A. (2014). On the Kumaraswamy-Kumaraswamy distribution. International Journal of Basic and Applied Sciences, **3**(4): 372-381.

Javanshiri, Z., Habibi Rad, A. and Arghami, N. R. (2015). Exp-Kumaraswamy distributions: Some properties and Applications. Journal of Sciences, **26**(1): 57-69.

Kenney, J. F. and Keeping, E. S. (1962). Mathematics of Statistics (3 ed.). Chapman & Hall Ltd, New Jersey.

Khan, M. S., King, R. and Hudson, I. L. (2016). Transmuted kumaraswamy distribution.Statistics in Transition, **17**(2): 183-210.

Kumaraswamy, P. (1980). Generalized probability density-function for double-bounded random processes. Journal of Hydrology, **46**: 79–88.

Lomax, K.S. (1954). Business failures: Another example of the analysis of failure data, Journal of the American Statistical Association, **49**: 847–852.

Moors, J. J. (1988). A quantile alternative for kurtosis. Journal of the Royal Statistical Society: Series D, **37**: 25–32.

Nadarajah, S. and Eljabri, S. (2013). The Kumaraswamy GP distribution. Journal of Data Science, **11**: 739-766.

Ramos, M. W., Marinho, P. R. D., Cordeiro, G. M., Silva, R. V. and Hamedani, G. (2015). The Kumaraswamy-G Poisson family of distributions. Journal of Statistical Theory and Applications, **14**(3): 222-239.

Shaw, W. T. and Buckley, I. R. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map.