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Stochastic Modeling of COVID-19: A SEI_AI_SQR-type model

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Abstract. This paper presents a model for the Corona-Virus (COVID-19) disease taking into account random perturbations. The proposed model is composed of six different classes namely the Susceptible population, the Exposed population, the Asymptomatic infectious population, the Symptomatic Infectious population, the Quarantined population and the Recovered population (SEI_AI_SQR). Using appropriately formulated stochastic Lyapunov functions, we established sufficient conditions for the existence and uniqueness of the positive solutions to the model. The condition for the extinction of the disease is also established. Numerical simulations are applied to illustrate the analytical results obtained herein.The reproduction number was obtained as $R_0^S = 0.2585 < 1$ and $R_0^S = 2.4423 > 1$ which show that the stability analysis of the equilibrium point is locally asymptotically stable whenever the basic reproduction number $R_0^S < 1$ and unstable whenever $R_0^S > 1$.

Keywords: Stochastic Model, COVID-19, SEI_AI_SQR Model, Lyapunov Function, Milstein's Higher Order Method

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1. Introduction

The 2019 novel coronavirus has been known to the virologist's community as Severe Acute Respiratory Syndrome Coronavirus-2 (SARS-CoV-2) (Lai *et al.*, 2020). Corona virus is a family of many diverse and numerous viruses that can infect both humans and animals that can cause a number of diseases (Mao, 1997). The name coronavirus, which means "crown virus" is related to the fact that all viruses of this family have a crown-like shape when observed under an electron microscope. The epidemic of novel coronavirus (COVID-19) infections that began in China in late 2019 has rapidly grown and cases have

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been reported worldwide. The virus appears to be transferred mostly through narrow respiratory droplets by coughing, sneezing, or people's interaction in close proximity (usually less than one metre) with each other for a certain time frame. These droplets can further be inhaled or can stay on the surfaces that came in contact with the infected person that can now cause infection in others by touching their nose, mouth or eyes. The virus possesses the ability to survive on various surfaces commencing several hours (e.g. copper, cardboard), up to a few days (e.g. plastic and stainless steel). Nonetheless, the quantity of the viable virus certainly decays over a time span and might not be present in sufficient quantity for causing the infection (Din *et al.,* 2020). The COVID-19 pandemic is considered as the biggest global threat worldwide because of thousands of confirmed infections, accompanied by thousands of deaths over the world. The only way to stop the spread of this disease is to quarantine or isolate the initially infected population as shown by the Chinese Government and adhering to the safety guidelines of World Health Organisation (WHO). Most of the real world phenomenon are not simply deterministic, because in deterministic models, the output of the model is fully determined by the parameter values and the initial conditions. Stochastic models possess some inherent randomness. The same set of parameter values and initial conditions will lead to an ensemble of different outputs. A stochastic model includes a random component that uses a distribution as one of the inputs, and results in a distribution for the output. These distributions may reflect the uncertainty in what the input should be (e.g. a deterministic input plus noise), or may reflect a random process (i.e. a stochastic input) (Perko, 2013, Wu and McGoogaan, 2020). In this paper, we shall propose a stochastic epidemic model of COVID-19 virus with a varying population environment and categorize the total population into six different classes. The first class is the susceptible individuals with white noise. The second class includes the exposed individual with white noise. The third is the asymptomatic infected individuals with white noise. The fourth is the symptomatic infected individual with white nose. The fifth class consists of the quarantine individuals with white noise. The sixth class consists of the recovered individuals with white noise. The existence and uniqueness of the positive solution of the proposed model and the disease' extinction for the COVID-19 are carefully discussed. Furthermore, we simulate the solution of the proposed model by using the higher-order stochastic Milstein method, (Higham, 2001).

2. Materials and Method

2.1 *Preliminaries*

Definitions

Mao (1997) (i) The triple (Ω, f, P) is called complete probability space if f contains all subsets G of Ω with P-outer measure zero, that is, with

$$P*(G)\inf P(F); F \in f, G \subset F = 0 \tag{1}$$

The subsets F of the set Ω which belong to F are called F-measurable sets. These sets are also called events. For instance, $P(F)$ = "The probability that the event F occurs". Particularly, if $P(F) = 1$, then we say that "F occurs with

probability 1" or "almost surely (a.s)"
(ii) Let (Ω, f, P) be a complete probability space. A random variable X is an f measurable function $X : \Omega \to R^n$. Every random variable induces a probability measure on R^n.
(iii) A stochastic process defined on a probability space (Ω, f, P) is a parameterized collection of random variables $X_t t \in T$ with index (or parameter) space T and assuming values in R^n. note that for each $t \in T$, fixed, we have a random variable $X_t : \omega \to R^n; \omega \in \Omega$. On the other hand, fixing $\omega \in \Omega$ we have the sample path/trajectory or a realization of the stochastic process: $X_t : t \to R^n; t \in T$
(iv) Let (Ω, f, P) be a probability space. If X is a real valued random variable and is integrable with respect to the probability measure P, then the expectation of X (with respect to P) is defined as:

$$E(X) = \int X(\omega)dP(\omega), \omega \in \Omega \tag{2}$$

(v) let $X_t t \in T$ be a real-valued stochastic process with discrete or continuous index set T. then $X_t t \in T$ is called a Martingale if expectation, $E[|X_n|] < \infty, \quad \forall \quad t \in T$ and if the conditional expectation is given by

$$E[X_t n + 1 | X_t 1 = x1, X_t 2 = x2, ..., X_t n = xn] = xn \tag{3}$$

Equivalently, $E[X_t n + 1 | f_t n] = xn$
(vi) Let (Ω, f, P) be a probability space with filtration $\{f_t\}_{t \geq 0}$. A one dimensional Brownian motion is a real-valued continuous $\{f_t\}$-adapted process $\{B_t\}_{t \geq 0}$ with the following properties:

(i) $B_0 = 0 a.s$
(ii) for $0 \leq S < t < \infty$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$.
(iii) for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is independent of f_s
(iv) for almost every $\omega \in \Omega$, the Brownian sample path $B(\omega)$ is no where differentiable.
(v) $\{B_t\}_t \geq 0$ is a continuous square-integrable Martingale and its quadratic variation $\langle B_t B \rangle_t = t$.

The following notations are introduced
$a \bigvee b =$ the maximum of a and b
$a \bigwedge b =$ the minimum of a and b
$R_+ =$ the set of all non-negative real numbers , that is $R_+ = [0, \infty)$
$R^d =$ The d-dimensional Euclidean space.
$R^d_+ = \{x = (x_1, ..., x_d) \in R^d; x_i > 0, 1 \leq i \leq d\}$,
$L^p([a, b]; R^d) is$ the family of R^d-valued f_t-adapted processes $\{f(t)\}_{a \leq t \leq b}$ such that

$$\int_a^b |f(t)|^p dt < \infty \tag{4}$$

$C^2\ (R_+ \times R^d; R_+)$ is the family of all non-negative real- valued functions $V(t,x)$ defined on R_+R^d such that they are twice continuously differentiable in x and once in t.
Let $x(t)$ be a one-dimensional Ito process on $t \geq 0$ with the stochastic differential

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB_t, x(0) = x_0 \tag{5}$$

where $f \in L^1\ (R_+ \times R; R)$ and $g \in L^2\ (R_+ \times R; R)$.
Let $V \in C^2\ (R_+ \times R; R)$. Then $V(t, x(t))$ is an Ito process with the stochastic differential given by:

$$\begin{aligned} dV(t,x(t)) =&[V_t(t,x(t)) + V_x(t,x(t))f(t,x(t)) + \frac{1}{2}V_{xx}(t,x(t))g^2(t,x(t))]dt \\ &+ V_x(t,x(t))g(t,x(t))dB_t \end{aligned} \tag{6}$$

Let $x(t)$ be a d-dimensional Ito process on $t \geq 0$ with the stochastic differential

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB_t, x(0) = x_0 \tag{7}$$

where $f \in L^1(R_+ \times R; R)$ and $g \in L^2(R_+ \times R^d; R^{d\times m})$.
Let $V \in C^2(R_+ \times R^d; R_+)$. Then $V(t, x(t))$ is again an Ito process with the stochastic differential given by:

$$\begin{aligned} dV(t,x(t)) =&[V_t(t,x(t)) + V_x(t,x(t))f(t,x(t)) + \frac{1}{2}trace(g^T(t,x(t)))V_{xx}(t,x(t))g(t,x(t))]dt \\ &+ V_x(t,x(t))g(t,x(t))dB_t \end{aligned} \tag{8}$$

The infinitesimal generator L associated with system (6) is defined by Mao (1997): as:

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d}[g^T(x,t)g(x,t)]_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} \tag{9}$$

Theorem according to Mao (1997), (i) if L acts on a function $V \in C^2(R_+ \times R^d; R_+)$.

$$LV = [V_t(t,x(t)) + V_x(t,x(t))f(t,x(t)) + \frac{1}{2}trace[g^T(t,x(t))V_{xx}(t,x(t))g(t,x(t))]dt \tag{10}$$

where $V_t = \frac{\partial V}{\partial t}, V_x = (\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_d}), V_x x = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{d\times d}$.

Again, the one dimensional Ito's lemma in Mao (1997) can be re-written as:

$$dV(x(t)) = LV(t, x(t))dt + V_x(t, x(t))g(t, x(t))dB_t \tag{11}$$

(ii) let $f, g \in M^p([a,b]; R^d)$ and let α, β be two real numbers. Then

(i) $\int_a^b f(t)dB(t)$ is f_t - measurable
(ii) $E\int_a^b f(t)dB(t) = 0$
(iii) $E\int_b^a |f(t)dB(t)|^2 = E\int_b^a |f(t)|^2 dt.$
(iv) $\int_b^a |as(t) + \beta g(t)dB(t) = \alpha \int_b^a f(t)dB(t) + \beta \int_b^a g(t)dB(t)$

(iii) Consider the stochastic differential equation

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB_t, x(t_0) = x_0, t_0 \leq t \leq T. \tag{12}$$

Assume that there exists two positive constants K_1 and K_2 such that for all $x, y \in R^d, t \in [t_0, T]$

- (Lipschitz condition) :
  $\forall \quad x, y \in R^d, t \in [t_0, T], |f(x,t) - f(t,y)|^2 V |g(t,x) - g(t,y)|^2 \leq K_1|x - y|$
- (Linear growth condition):
  $\forall \quad (t, x) \in [t_0, T] X R^d, |f(t,x)|^2 V |g(t,y)|_2^2 (1 + |x|^2).$

Then there exists a unique local solution.

2.2 *Model Formulation*

In this section, we formulate a stochastic model to study the transmission dynamics of COVID-19. According to the characteristics of the disease, we propose a Susceptible-Exposed-Asymptomatic Infectious-Symptomatic Infectious-Quarantined-Removed epidemic model. We take into consideration the variations of the population environment in order to study the dynamics of COVID-19, in particular its long-term behaviour. Some of the assumptions underlying the formulation of the model are:

1 The total population at any time t is denoted by $N(t)$ and it is classified into six exclusive groups of individuals: the Susceptible class $S(t)$, the Exposed class $E(t)$, the Asymptomatic infectious class $I_A(t)$, the Symptomatic infectious class $I_S(t)$, the Quarantine class $Q(t)$, and the Recovered $R(t)$. That is, $S(t) + E(t) + I_A(t) + I_S(t) + Q(t) + R(t) = N(t)$ which is changing with time t.
2 The state variables and parameters included in the model are assumed to be non-negative.
3 The infected individuals move to the quarantined class
4 Once the infection is confirmed, then the quarantined will go back to the infected compartment.

In the light of the assumptions, we obtain the deterministic model:

$$
\left.\begin{aligned}
\frac{dS}{dt} &= \pi - \rho_1\beta_A I_A S - \rho_2\beta_S I_S S - \mu S\\
\frac{dE}{dt} &= \rho_1\beta_A I_A S + \rho_2\beta_S I_S S - (\alpha+\mu)E\\
\frac{dI_A}{dt} &= (1-\delta)\alpha E - (\gamma+\mu_x+\mu)I_A\\
\frac{dI_S}{dt} &= \alpha\delta E - (\omega+\mu_x+\mu)I_S\\
\frac{dQ}{dt} &= \gamma I_A + \omega I_S - (\theta+\mu_x+\mu)Q\\
\frac{dR}{dt} &= \theta Q - \mu R
\end{aligned}\right\} \tag{13}
$$

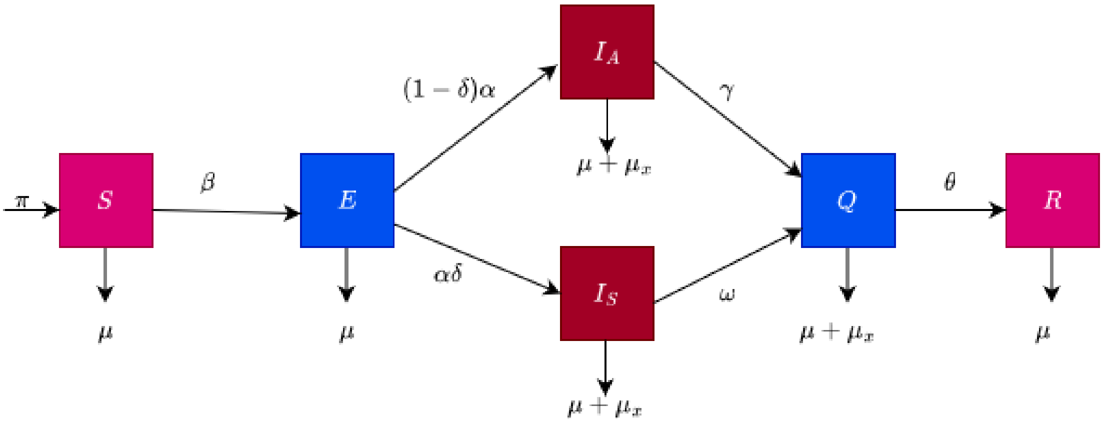

Figure 1: The schematic diagram of the SEI_AI_SQR Model

where π is the recruitment rate into the susceptible class and μ, is the natural death rate, β is the force of infection, ρ_1 and ρ_2 are the effective contact rate between the symptomatic and asymptomatic individuals respectively, β_A and β_S are the transmission probabilities of the virus from infectious individuals to the susceptible class, μ_x is death due to COVID-19, $(1-\delta)\alpha$ the proportion of individuals moving into the asymptomatic class, $\alpha\delta$ the rate at which the individuals move into the symptomatic class, γ the rate at which the asymptomatic class move into the quarantine class, ω is the rate at which the symptomatic class are quarantined and θ is the recovery rate.
From the deterministic model in (13), we then formulate the stochastic model:

$$
\left.\begin{aligned}
dS(t) &= (\pi - \rho_1\beta_A I_A(t)S(t) - \rho_2\beta_S I_S(t)S(t) - \mu S(t))dt + \sigma_1 S(t)dW_1(t)\\
dE(t) &= (\rho_1\beta_A I_A(t)S(t) + \rho_2\beta_S I_S(t)S(t) - (\alpha+\mu)E(t))dt + \sigma_2 E(t)dW_2(t)\\
dI_A(t) &= (1-\delta)\alpha E(t) - (\gamma+\mu_x+\mu)I_A(t))dt + \sigma_3 I_A(t)dW_3(t)\\
dI_S(t) &= (\alpha\delta E(t) - (\omega+\mu_x+\mu)I_S(t))dt + \sigma_4 I_S(t)dW_4(t)\\
dQ(t) &= (\gamma I_A(t) + \omega I_S(t) - (\theta+\mu_x+\mu)Q(t))dt + \sigma_5 Q(t)dW_5(t)\\
dR(t) &= (\theta Q(t) - \mu R(t))dt + \sigma_6 R(t)dW_6(t)
\end{aligned}\right\} \tag{14}
$$

where $W_1(t)$, $W_2(t)$, $W_3(t)$, $W_4(t)$, $W_5(t)$, $W_6(t)$ are independent standard Brownian motions, and σ_1, σ_2, σ_3, σ_4, σ_5, σ_6 as the intensities of the standard Gaussian white noises respectively.
The terms $\sigma_1 S(t)dW_1(t)$, $\sigma_2 E(t)dW_2(t)$, $\sigma_3 I_A(t)dW_3(t)$, $\sigma_4 I_S(t)dW_4(t)$, $\sigma_5 Q(t)dW_5(t)$, $\sigma_6 R(t)dW_6(t)$ are the interactions between the individuals and

the environment. The initial conditions are given by
$(S(0), E(0), I_A(0), I_S(0), Q(0), R(0)) \in R_+^6$.

2.3 *Existence and Uniqueness of the Solution of Stochastic Model*

This section provides the existence and uniqueness of solution of the proposed stochastic model in (14) using the Lyapunov method.

Theorem 2:
For any given initial values $X_0 = (S(0), E(0), I_A(0,)I_S(0), Q(0), R(0)) \in D$ there is a unique solution $X_t = X(t) = (S(t), E(t), I_A(t), I_S(t), Q(t), R(t))$ for $t \geq 0$, which will remain in D with probability one. That is, $P(S(t), E(t), I_A(t), I_S(t), Q(t), R(t)) \in D$ for all $t \geq 0$.

proof:
For any given initial value $X_0 = (S(0), E(0), I_A(0), I_S(0), Q(0), R(0)) \in D$ there is a unique local solution $X_t = X(t) = (S(t), E(t), I_A(t), I_S(t), Q(t), R(t))$ for $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that the solution is positive and exist globally, it is essential to establish that $\tau_e = +\infty$ almost surely $(a.s.)$. Suppose that $k_0 > 0$ be sufficiently large such that $S(0), E(0), I_A(0,)I_S(0), Q(0), R(0)$ stays within $[\frac{1}{k_o}, k_0]$. Then, each $k \geq k_0$, define the stopping time
$\tau_k = \inf\{t \in [0, \tau_e) : S(t) \notin (\frac{1}{k}, k)$ or $E(t) \notin (\frac{1}{k}, k)$ or $I_A(t) \notin (\frac{1}{k}, k)$ or $I_S(t) \notin (\frac{1}{k}, k)$ or $Q(t) \notin (\frac{1}{k}, k)$ or $R(t) \notin (\frac{1}{k}, k)\}$
where $\inf \phi = \infty$ and ϕ denotes the empty set. Obviously, τ_k is increasing as $k \to \infty$, thus $\tau_\infty = \lim_{k\to\infty} \tau_k$, so, $\tau_\infty \leq \tau_k$ a.s. If we prove that $\tau_\infty = \infty$ a.s then $\tau_e = \infty$ and $((S(t), E(t), I_A(t), I_S(t), Q(t), R(t)) \in D$ a.s for all $t \geq 0$. Assume that $t_\infty < \infty$, then there exist a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \epsilon$. Thus there is an integer $k_1 \geq k_0$ such that

$$P\tau_k \leq T > \epsilon \quad for \quad all \quad k \geq k_1 \tag{15}$$

Define C^2 function $V_1 : R_+^6 \to R_+$ by

$$\begin{aligned} V_1(S, E, I_A, I_S, Q, R) =& (S - 1 - \ln S) + (E - 1 - \ln E) + (I_A - 1 - \ln I_A) + (I_S - 1 - \ln I_S) \\ &+ (Q - 1 - \ln Q) + (R - 1 - \ln R) \end{aligned} \tag{16}$$

Applying Ito's Lemma on (16) gives
$dV_1 = (1 - \frac{1}{S})dS + \frac{1}{2}\frac{1}{S}^2(dS)^2 + (1 - \frac{1}{E})dE + \frac{1}{2}\frac{1}{E}^2(dE)^2 + (1 - \frac{1}{I_A})dI_A + \frac{1}{2}\frac{1}{I_A}^2(dI_A)^2 + (1 - \frac{1}{I_S})dI_S + \frac{1}{2}\frac{1}{I_S}^2(dI_S)^2 + (1 - \frac{1}{Q})dQ + \frac{1}{2}\frac{1}{Q}^2 + (1 - \frac{1}{R})dR + \frac{1}{2}\frac{1}{R}^2(dR)^2$
$= (1 - \frac{1}{S})(\pi - \rho_1\beta_A I_A S - \rho_2\beta_S I_S S - \mu S)dt + (1 - \frac{1}{S})\sigma_1 S(t)dW_1(t) + \frac{1}{2}\sigma_1^2 dt +$
$(1 - \frac{1}{E})(\rho_1\beta_A I_A S + \rho_2\beta_S I_S S - (\alpha + \mu)E)dt + (1 - \frac{1}{E})\sigma_2 E(t)dW_2(t) + \frac{1}{2}\sigma_2^2 dt +$
$(1 - \frac{1}{I_A})((1 - \delta)\alpha E - (\gamma + \mu_x + \mu)I_A)dt + (1 - \frac{1}{I_A})\sigma_3 I_A(t)dW_3(t) + \frac{1}{2}\sigma_3^2 dt +$
$(1 - \frac{1}{I_S})(\alpha\delta E - (\omega + \mu_x + \mu)I_S)dt + (1 - \frac{1}{I_S})\sigma_4 I_S(t)dW_4(t) + \frac{1}{2}\sigma_4^2 dt +$

$(1-\frac{1}{Q})(\gamma I_A+\omega I_S-(\theta+\mu_x+\mu)Q)dt+(1-\frac{1}{Q})\sigma_5 Q(t)dW_5(t)+\frac{1}{2}\sigma_5^2 dt+$
$(1-\frac{1}{R})(\theta Q-\mu R)dt+(1-\frac{1}{Q})\sigma_6 Q(t)dW_6(t)+\frac{1}{2}\sigma_6^2 dt$

Which simplifies to

$$\begin{aligned}dV_1 &= [(1-\tfrac{1}{S})(\pi-\rho_1\beta_A I_A S-\rho_2\beta_S I_S S-\mu S)+\tfrac{1}{2}\sigma_1^2+\\&(1-\tfrac{1}{E})(\rho_1\beta_A I_A S+\rho_2\beta_S I_S S-(\alpha+\mu)E)+\tfrac{1}{2}\sigma_2^2+\\&(1-\tfrac{1}{I_A})((1-\delta)\alpha E-(\gamma+\mu_x+\mu)I_A)+\tfrac{1}{2}\sigma_3^2+\\&(1-\tfrac{1}{I_S})(\alpha\delta E-(\omega+\mu_x+\mu)I_S)+\tfrac{1}{2}\sigma_4^2+\\&(1-\tfrac{1}{Q})(\gamma I_A+\omega I_S-(\theta+\mu_x+\mu)Q)+\tfrac{1}{2}\sigma_5^2+\\&(1-\tfrac{1}{R})(\theta Q-\mu R)+\tfrac{1}{2}\sigma_6^2]dt+\\&(\sigma_1(S-1)dW_1+\sigma_2(E-1)dW_2+\sigma_3(I_A-1)dW_3+\sigma_4(I_S-1)dW_4+\sigma_5(Q-1)dW_5+\sigma_6(R-1)dW_6)\end{aligned}$$

and hence we obtain that

$$dV_1=LV_1dt+(\sigma_1(S-1)dW_1+\sigma_2(E-1)dW_2+\sigma_3(I_A-1)dW_3+\sigma_4(I_S-1)dW_4+\sigma_5(Q-1)dW_5+\sigma_6(R-1)dW_6)$$

where

$$\begin{aligned}LV_1 &= (1-\tfrac{1}{S})(\pi-\rho_1\beta_A I_A S-\rho_2\beta_S I_S S-\mu S)+\tfrac{1}{2}\sigma_1^2+\\&(1-\tfrac{1}{E})(\rho_1\beta_A I_A S+\rho_2\beta_S I_S S-(\alpha+\mu)E)+\tfrac{1}{2}\sigma_2^2+\\&(1-\tfrac{1}{I_A})((1-\delta)\alpha E-(\gamma+\mu_x+\mu)I_A)+\tfrac{1}{2}\sigma_3^2+\\&(1-\tfrac{1}{I_S})(\alpha\delta E-(\omega+\mu_x+\mu)I_S)+\tfrac{1}{2}\sigma_4^2+\\&(1-\tfrac{1}{Q})(\gamma I_A+\omega I_S-(\theta+\mu_x+\mu)Q)+\tfrac{1}{2}\sigma_5^2+\\&(1-\tfrac{1}{R})(\theta Q-\mu R)+\tfrac{1}{2}\sigma_6^2\end{aligned}$$

$$\left.\begin{aligned}&LV_1=\pi-\mu S-\mu E-\mu I_A-\mu I_S-\mu Q-\mu R\\&+(-\tfrac{\pi}{S}+\rho_1\beta_A I_A+\rho_2\beta_S I_S+\mu)+(-\tfrac{\rho_1\beta_A I_A S}{E}-\tfrac{\rho_2\beta_S I_S S}{E}+\alpha+\mu)+(-\tfrac{\alpha E}{I_A}\\&+\tfrac{\delta\alpha E}{I_A}+\gamma+\mu_x+\mu)+(-\tfrac{\alpha\delta E}{I_S}+\omega+\mu_x+\mu)\\&+(-\tfrac{\gamma I_A}{Q}-\tfrac{\omega I_S}{Q}+\theta+\mu_x+\mu)+(-\tfrac{\theta Q}{R}+\mu)+\tfrac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2}{2}\\&LV_1\le\pi-\mu N-\mu_x(I_A+I_S+Q)+\mu+(\alpha+\mu)+(\gamma+\mu_x+\mu)+(\omega+\mu_x+\mu)\\&+(\theta+\mu_x+\mu)+\mu+(\rho_1\beta_A I_A+\rho_2\beta_S I_S)+\\&\tfrac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2}{2}\end{aligned}\right\}\quad(17)$$

$$LV_1\le\pi+6\mu+3\mu_x+\rho_1\beta_A I_A+\rho_2\beta_S I_S+\gamma+\omega+\theta+\alpha+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2}{2}$$

Note that $S+E+I_A+I_S+Q+R\le\frac{\pi}{\mu}$, then

$$LV_1\le\pi+6\mu+3\mu_x+\gamma+\omega+\theta+\alpha+\rho_1\beta_A I_A+\rho_2\beta_S I_S+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2}{2}=:K$$

Thus we have

$$\begin{aligned}dV_1=&Kdt+(\sigma_1(S-1)dW_1+\sigma_2(E-1)dW_2+\sigma_3(I_A-1)dW_3+\sigma_4(I_S-1)dW_4+\sigma_5(Q-1)dW_5\\&+\sigma_6(R-1)dW6)\end{aligned}\quad(18)$$

Integrating both sides of (18) from 0 to $\tau_k \bigwedge T$, we obtain

$$\left.\begin{array}{l} \int_0^{\tau_k \bigwedge T} dV_1(S(r), E(r), I_A(r), I_S(r), Q(r), R(r)) \\ \leq \int_0^{\tau_k \bigwedge T} K dr + \int_0^{\tau_k \bigwedge T} (\sigma_1(S(r)-1)dW_1(r) + \sigma_2(E(r)-1)dW_2(r) + \sigma_3(I_A(r)-1)dW_3(r) + \\ \sigma_4(I_S(r)-1)dW_4(r) + \sigma_5(Q(r)-1)dW_5(r) + \sigma_6(R(r)-1)dW_6(r)) \end{array}\right\} \quad (19)$$

Taking expectation of both sides of (19) and Recalling $E\int_0^b f(t)dB(t) = 0$, we have
$EV_1(S(\tau_k \bigwedge T), E(\tau_k \bigwedge T), I_A(\tau_k \bigwedge T), I_S(\tau_k \bigwedge T), Q(\tau_k \bigwedge T), R(\tau_k \bigwedge T))$
$\leq V_1(S(0), E(0), I_A(0), I_S(0), Q(0), R(0) + E\int_0^{\tau_k \bigwedge T} K dr,$
$EV_1(S(\tau_k \bigwedge T), E(\tau_k \bigwedge T), I_A(\tau_k \bigwedge T), I_S(\tau_k \bigwedge T), Q(\tau_k \bigwedge T), R(\tau_k \bigwedge T))$
$\leq V_1(S(0), E(0), I_A(0), I_S(0), Q(0), R(0) + KT$

Let $\Omega_k = \{\tau_k \leq T\}$ for all $k \geq k_1$ then by (3) we have $P(\Omega_k) \geq \epsilon$.

Note that for every $\omega \in \Omega_k$, we have at least $S(\tau_k, \omega)$ or $E(\tau_k, \omega)$ or $I_A(\tau_k, \omega)$ or $I_S(\tau_k, \omega)$ or $Q(\tau_k, \omega)$ or $R(\tau_k, \omega)$ which is equivalent to
k or $\frac{1}{k}$, thus $V(S(\tau_k, \omega)$ or $E(\tau_k, \omega)$ or $I_A(\tau_k, \omega)$ or $I_S(\tau_k, \omega)$ or $Q(\tau_k, \omega)$ or $R(\tau_k, \omega)$ is no less than
$k - 1 - \ln k$ or $\frac{1}{k} - 1 - \ln\frac{1}{k} = \frac{1}{k} - 1 + lnk$

Hence,
$V(S(\tau_k, \omega), E(\tau_k, \omega), I_A(\tau_k, \omega), I_S(\tau_k, \omega), Q(\tau_k, \omega), R(\tau_k, \omega)) \geq (k - 1 - lnk) \bigwedge(\frac{1}{k} - 1 + \ln k)$

Then it follows that,
$V_(S(0), E(0), I_A(0), I_S(0), Q(0), R(0) + KT$
$\geq E(1\Omega_k V_1(S(\tau_k \bigwedge T), E(\tau_k \bigwedge T), I_A(\tau_k \bigwedge T), I_S(\tau_k \bigwedge T), Q(\tau_k \bigwedge T), R(\tau_k \bigwedge T)),$
$= E(1\Omega_k(\omega)V_1(S(\tau_k, \omega), E(\tau_k, \omega), I_A(\tau_k, \omega), I_S(\tau_k, \omega), Q(\tau_k, \omega), R(\tau_k, \omega)),$
$\geq E(1\Omega_k(\omega)(k - 1 - \ln k) \bigwedge(\frac{1}{k} - 1 + \ln k)), = (k - 1 - \ln k) \bigwedge(\frac{1}{k} - 1 + \ln k)E(1\Omega_k(\omega),$

$$\geq \epsilon(k - 1 - \ln k) \bigwedge(\frac{1}{k} - 1 + \ln k) \quad (20)$$

where 1Ω is the indicator function of $\Omega_k(\omega)$. if $k \to \infty$ then
$\infty > V(S(0), E(0), I_A(0), I_S(0), Q(0), R(0)) + KT = \infty$ becomes a contradiction, thus, the only possibility is that $\tau_\infty = \infty$, which completes the proof.

2.4 *Extinction of the disease*

In modeling the dynamics of any infectious disease, it is important to study conditions under which the disease will go into extinction or die out from the population. In this section, we will show that if the white noise is sufficiently large, then the solution of associated stochastic system (14) will become extinct with probability one. Let us consider the following notation and results:

$\langle X(t)\rangle = \frac{1}{t}\int_0^t X(\tau)d\tau$

Lemma 1: (Strong law of large number, Mao (1997)) Let $M = \{M_t\}$ $t \geq 0$ be continuous and real valued local martingale vanishing at $t = 0$ and $\langle M, M\rangle\, t$ be its quadratic variation. Then

$$\lim_{t\to\infty} \langle M, M\rangle\, t = \infty, a.s, \quad \lim_{t\to\infty} \frac{M_t}{\langle M, M\rangle\, t} = 0 \quad a.s \tag{21}$$

Also $\lim_{t\to\infty} \sup \frac{\langle M,M\rangle t}{t} < 0$ a.s, $\lim_{t\to\infty} \frac{M_t}{t} = 0$ a.s.

Lemma 2: Let $(S(t), E(t), I_A(t), I_S(t), Q(t), R(t))$ be the solution of the stochastic system (14) with initial value $(S(0), E(0), I_A(0), I_S(0), Q(0), R(0)) \in R^6_+$ a.s then

$$\lim_{t\to+\infty} \frac{(S(t), E(t), I_A(t), I_S(t), Q(t), R(t))}{t} = 0 \tag{22}$$

Moreover,
If $\mu > (\frac{\sigma_1^2 V \sigma_2^2 V \sigma_3^2 V \sigma_4^2 V \sigma_5^2 V \sigma_6^2}{2})$ then

$$\left.\begin{array}{l} \lim_{t\to+\infty} \frac{1}{t}\int_0^t S(\tau)dW_1(\tau) = 0, \lim_{t\to+\infty} \frac{1}{t}\int_0^t E(\tau)dW_2(\tau) = 0, \lim_{t\to+\infty} \frac{1}{t}\int_0^t I_A(\tau)dW_3(\tau) = 0, \\ \lim_{t\to+\infty} \frac{1}{t}\int_0^t I_S(\tau)dW_4(\tau) = 0, \lim_{t\to+\infty} \frac{1}{t}\int_0^t Q(\tau)dW_5(\tau) = 0, \lim_{t\to+\infty} \frac{1}{t}\int_0^t R(\tau)dW_6(\tau) = 0 \end{array}\right\} \tag{23}$$

the threshold quantity R_0^S for the stochastic system (14) can be written as

$$\left.\begin{array}{l} R_1^S = \frac{(\rho_1\beta_A\frac{\pi}{\mu})(1-\delta)\alpha}{(\alpha+\mu)(\gamma+\mu_x+\mu)} - \frac{\delta_3^2}{2(\gamma+\mu_x+\mu)} \\ R_2^S = (\frac{\rho_2\beta_S\frac{\pi}{\mu})\delta\alpha}{(\alpha+\mu)(\omega+\mu_x+\mu)} - \frac{\delta_4^2}{2(\omega+\mu_x+\mu)} \end{array}\right\} \tag{24}$$

The following theorem gives the necessary conditions for the extinction of infections. theorem 3 For any given initial value $(S(0), E(0), I_A(0), I_S(0), Q(0), R(0)) \in R^6$ the solution $(S(t)$, $E(t)$, $I_A(t)$, $I_S(t)$, $Q(t)$, $R(t))$ of the system in (14) has the following properties: if

a (i) $\sigma_3^2 > \frac{((\rho_1\beta_A\frac{\pi}{\mu})(1-\delta)\alpha)^2}{2(\alpha+\mu)^2(\gamma+\mu_x+\mu)}$, then Covid-19 of I_A goes into extinction a.s

(ii) $\sigma_4^2 > \frac{((\rho_2\beta_S\frac{\pi}{\mu})\delta\alpha)^2}{2(\alpha+\mu)^2(\omega+\mu_x+\mu)}$, then Covid-19 of I_S goes into extinction a.s

b (i) $R_1^S < 1$, then I_A dies out with probability 1

(ii) $R_2^S < 1$, then I_S dies out with probability 1

This implies that , if condition (a) and (b) hold, then
$\lim_{t\to\infty} \frac{\langle logI_A\rangle}{t} < 0$ and $\lim_{t\to\infty} \frac{\langle logI_S\rangle}{t} < 0$ a.s
That is, the disease goes into extinction with probability 1.

Moreover,

$$\lim_{t\to\infty}\langle S(t)\rangle=\frac{\pi}{\mu},\lim_{t\to\infty}\langle E(t)\rangle=0,\lim_{t\to\infty}\langle I_A(t)\rangle=0,\lim_{t\to\infty}\langle I_S(t)\rangle=0,\lim_{t\to\infty}\langle Q(t)\rangle=0,\lim_{t\to\infty}\langle R(t)\rangle=0 \tag{25}$$

Proof: Integrating the model in (14), we have

$$\left.\begin{array}{l}
\frac{S(t)-S(0)}{t}=\pi-\rho_1\beta_A\langle I_A(t)S(t)\rangle-\rho_2\beta_S\langle I_S(t)S(t)\rangle-\mu\langle S(t)\rangle+\frac{\delta_1}{t}\int_0^t S(\tau)dW_1(\tau),\\
\frac{E(t)-S(0)}{t}=\rho_1\beta_A\langle I_A(t)S(t)\rangle+\rho_2\beta_S\langle I_S(t)S(t)\rangle-(\alpha+\mu)\langle E(t)\rangle+\frac{\delta_2}{t}\int_0^t E(\tau)dW_2(\tau),\\
\frac{I_A(t)-I_A(0)}{t}=(1-\delta)\alpha\langle E(t)\rangle-(\gamma+\mu_x+\mu)\langle I_A(t)\rangle+\frac{\delta_3}{t}\int_0^t I_A(\tau)dW_3(\tau),\\
\frac{I_S(t)-I_S(0)}{t}=\alpha\delta\langle E(t)\rangle-(\omega+\mu_x+\mu)\langle I_S(t)\rangle+\frac{\delta_4}{t}\int_0^t I_S(\tau)dW_4(\tau),\\
\frac{Q(t)-Q(0)}{t}=\gamma\langle I_A(t)\rangle+\omega\langle I_S(t)\rangle-(\theta+\mu_x+\mu)\langle Q(t)\rangle+\frac{\delta_5}{t}\int_0^t Q(\tau)dW_5(\tau),\\
\frac{R(t)-R(0)}{t}=\theta\langle Q(t)\rangle+\mu\langle R(t)\rangle+\frac{\delta_6}{t}\int_0^t R(\tau)dW_6(\tau),
\end{array}\right\} \tag{26}$$

(a) if Ito's formula is applied to the third equation of (14), we have

$$dlogI_A(t)=[(1-\delta)\alpha E(t)-(\gamma+\mu_x+\mu)I_A(t)]\frac{1}{I_A}dt-\frac{\sigma_3^2}{2}dt+\sigma_3dW_3(t) \tag{27}$$

If we integrate the third equation of (26) within $[0,t]$, then we have

$logI_A(t)=\int_0^t(1-\delta)\alpha E(\tau)d\tau-(\gamma+\mu_x+\mu)t\int_0^t+\frac{\sigma_3^2}{2}dt+\frac{\sigma_3}{t}\int_0^t dW_3(\tau)+logI_A(0)$

$\leq\int_0^t(1-\delta)\alpha E(\tau)d\tau-(\gamma+\mu_x+\mu)t\int_0^t+\frac{\sigma_3^2}{2}dt+\frac{\sigma_3}{t}\int_0^t dW_3(\tau)+logI_A(0)$

$\leq\int_0^t(\frac{(1-\delta)\alpha\rho_1\beta_A\frac{\pi}{\mu}}{\alpha+\mu}-\frac{\sigma_3^2}{2})d\tau-(\gamma+\mu_x+\mu)t+\int_0^t\sigma_3dW_3(\tau)+logI_A(0)$

which can be re-written as

$logI_A(t)\leq-\frac{\sigma_3^2}{2}\int_0^t(1-\frac{(1-\delta)\alpha\rho_1\beta_A\frac{\pi}{\mu}}{\sigma_3^2(\alpha+\mu)})^2d(\tau)-(\gamma+\mu_x+\mu)t+\int_0^t(1-\frac{(1-\delta)\alpha\rho_1\beta_A\frac{\pi}{\mu}}{\sigma_3^2(\alpha+\mu)^2})^2d(\tau)+\int_0^t\sigma_3dW_3(\tau)+logI_A(0)$

$\leq-((\gamma+\mu_x+\mu)-\frac{((1-\delta)\alpha)^2(\rho_1+\beta_A\frac{\pi}{\mu})^2}{2\sigma_3^2(\alpha+\mu)^2}t\int_0^t\sigma_3dW_3(\tau)+logI_A$

Division by t gives

$$\frac{logI_A(t)}{t}\leq-((\gamma+\mu_x+\mu)-\frac{((1-\delta)\alpha)^2(\rho_1+\beta_A\frac{\pi}{\mu})^2}{2\sigma_3^2(\alpha+\mu)^2}+\frac{1}{t}\int_0^t\sigma_3dW_3(\tau)+\frac{logI_A(0)}{t} \tag{28}$$

By strong law of large number,

$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_3dW_3(\tau)=0$ a.s as $\sigma_3^2>\frac{((1-\delta)\alpha)^2(\rho_1+\beta_A\frac{\pi}{\mu}}{2(\alpha+\mu)^2(\gamma+\mu_x+\mu)}$,

by taking limit superior on both sides of (28), we obtain

$\lim_{t\to\infty}\sup\frac{logI_A(t)}{t}\leq-((\gamma+\mu_x+\mu)-\frac{((1-\delta)\alpha)^2(\rho_1+\beta_A\frac{\pi}{\mu})^2}{2\sigma_3^2(\alpha+\mu)})<0$

This implies that, $\lim_{t\to\infty}I_A(0)=0$

In a similar manner, it can be shown that

$\lim\limits_{t\to\infty}\sup\frac{logI_S(t)}{t} \leq -((\omega+\mu_x+\mu)-\frac{((\delta\alpha(\rho_2+\beta_S\frac{\pi}{\mu}))^2}{2\sigma_3^2(\alpha+\mu)}) < 0$
which implies that, $\lim\limits_{t\to\infty} I_S(0) = 0$

(b) Again if equation (27) is integrated within $[0,t]$ and divided by t, then we have

$$\left.\begin{aligned}
&\tfrac{logI_A(t)-logI_A(0)}{t} = (1-\delta)\alpha\left\langle E(t)\right\rangle - (\gamma+\mu_x+\mu)\tfrac{\sigma_3^2}{2}\tfrac{\sigma_3}{t}\int_0^t dW_3(\tau),\\
&\leq \tfrac{(1-\delta)\alpha(\rho_1\beta_A\frac{\pi}{\mu})}{(\alpha+\mu)} - (\gamma+\mu_x+\mu) - \tfrac{\sigma_3^2}{2}\tfrac{\sigma_3}{t}\int_0^t dW_3(\tau),\\
&= (\gamma+\mu_x+\mu)(\tfrac{(1-\delta)\alpha(\rho_1\beta_A\frac{\pi}{\mu})}{(\alpha+\mu)} - (\gamma+\mu_x+\mu - \tfrac{\sigma_3^2}{2(\gamma+\mu_x+\mu)} - 1) + \tfrac{\sigma_3}{t}\int_0^t dW_3(\tau),\\
&= (\gamma+\mu_x+\mu)(R_1^S-1) + \tfrac{\sigma_3}{t}\int_0^t dW_3(\tau),
\end{aligned}\right\} \tag{29}$$

Moreover, $M(t) = \frac{\sigma_3}{t}\int_0^t dW_3(\tau)$ is continuous (locally) and $M(0) = 0$ From Equation (14) and $t\to\infty$

$$\lim_{t\to\infty}\sup\frac{M(t)}{t} = 0 \tag{30}$$

If $R_1^S < 1$ then equation (29) becomes

$$\lim_{t\to\infty}\sup\frac{logI_A(t)}{t} \leq (\gamma+\mu_x+\mu)(R_1^S-1) < 0 \quad a.s \tag{31}$$

Equation (31) implies

$$\lim_{t\to\infty} I_A(t) = 0 \quad a.s \tag{32}$$

Likewise, if Ito's formula is applied to the fourth equation of (14), then we obtain

$$\begin{aligned}
\frac{logI_S(t)-logI_S(0)}{t} &\leq (\omega+\mu_x+\mu)(\frac{\alpha\delta(\rho_2\beta_S\frac{\pi}{\mu})}{(\alpha+\mu)(\omega+\mu_x+\mu)} - \frac{\sigma_4^2}{2(\omega+\mu_x+\mu)} - 1)\\
&+\frac{\sigma_4}{t}\int_0^t dW_4(\tau) = (\omega+\mu_x+\mu)(R_2^S-1)\frac{\sigma_4}{t}\int_0^t dW_4(\tau)
\end{aligned} \tag{33}$$

Note that $M(t) = \frac{\sigma_4}{t}\int_0^t dW_4(\tau)$, is also locally "continuous martingale" and by lemma (2)

$$and \quad t\to\infty, \quad we \quad have \quad \lim_{t\to\infty}\sup\frac{M(t)}{t} = 0 \tag{34}$$

$$\lim_{t\to\infty}\sup\frac{logI_S(t)}{t} \leq (\omega+\mu_x+\mu)(R_2^S-1) < 0 \quad a.s \tag{35}$$

Equation (35) gives

$$\lim_{t\to\infty} I_S(t) = 0 \quad a.s \tag{36}$$

3. Results and Discussion(Numerical Simulation)

The model in (14) is simulated in this section. The numerical scheme used is based on the Milstein's Higher Order Method (Higham, 2001). Numerical simulation is carried out to validate theoretical results discussed earlier. The initial conditions used for the simulation are assumed as follows: $(S(0) = 0.6, E(0) = 0.2, I_A(0) = 0.1, I_S(0) = 0.1, Q(0) = 0.5$ and $R(0) = 0.1$, the values of the parameters used are given in Table 1. In Figure 2, simulations of the system in (14) are presented when the contact rate and the stochastic white noise terms are respectively given by $\beta_S = 0.02, \beta_A = 0.02, \sigma_1 = 0.2, \sigma_2 = 0.5, \sigma_3 = 0.35, \sigma_4 = 0.1, \sigma_5 = 0.2, \sigma_6 = 0.4$, and the associated stochastic reproduction number is given by $R_0^S = 0.2585 < 1$. Figure 2 reveals that the disease will go into extinction exponentially with unit probability. This also confirms the result of Theorem 2.

In Figure 3, simulations of the system in (14) are presented, and the parameter values and the white noise terms are respectively given by $\mu = 0.01, \pi = 0.9, \beta_S = 0.09, \beta_A = 0.08, \delta = 0.05, \alpha = 0.5, \mu_x = 0.2, \gamma = 0.3, \rho_1 = 0.8, \rho_2 = 0.7, \omega = 0.4, \theta = 0.1, \sigma_1 = 0.98, \sigma_2 = 0.8, \sigma_3 = 0.74, \sigma_4 = 0.9, \sigma_5 = 0.6, \sigma_6 = 0.45$ and the associated stochastic reproduction number is given by $R_0^S = 2.4423 > 1$. Figure 3 reveals that the disease will persist.

Table 1: Parameter Values

Notation	Values	Reference
S	0.6	Ding *et al.* (2021)
E	0.2	Ding *et al.* (2021)
I_A	0.1	Ding *et al.* (2021)
I_S	0.1	Ding *et al.* (2021)
Q	0.5	Din *et al.* (2020)
R	0.1	Ding *et al.* (2021)
π	0.9	assumed
α	0.5	Din *et al.* (2020)
μ_x	0.2	Din *et al.* (2020)
γ	0.01	Ding *et al.* (2021)
ω	0.4	Assumed
δ	0.05	Ding *et al.* (2021)
θ	0.1	Din *et al.* (2020)

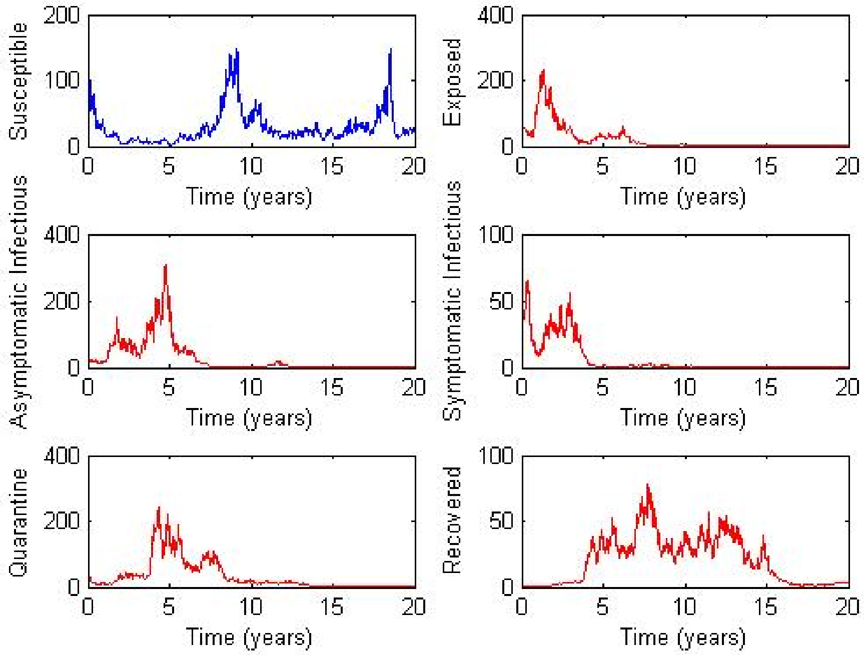

Figure 2: Simulation for the stochastic model when $\sigma_1 = 0.2, \sigma_2 = 0.5, \sigma_3 = 0.35, \sigma_4 = 0.1, \sigma_5 = 0.2, \sigma_6 = 0.4$, and the associated stochastic reproduction number is given by $R_0^S = 0.2585 < 1$

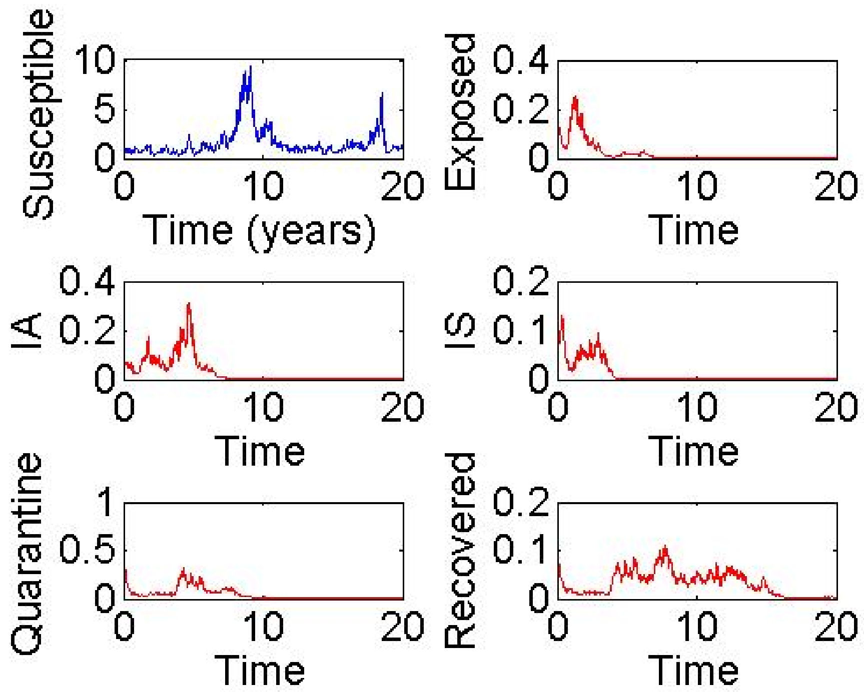

Figure 3: Simulation for the stochastic model when $\sigma_1 = 0.98, \sigma_2 = 0.8, \sigma_3 = 0.74, \sigma 4{=}0.9, \sigma_5 = 0.6, \sigma_6 = 0.45$ and the associated stochastic reproduction number is given by $R_0^S = 2.4423 > 1$.

The simulation of the stochastic model (14) is displayed in Figures 1 and 2 in order to evaluate the effects of stochastic white noise intensities. Increasing the white noise intensities σ_1, σ_2, σ_3, σ_4, σ_5, and σ_6 are observed to accelerate the process of extinction. Additionally, it demonstrates that persistent efforts to increase stochastic disruptions through quarantine could significantly decrease the movement and spread of COVID-19.

4. Conclusion

In this work, we have presented a stochastic model for COVID-19. The existence and the uniqueness of global solution of the stochastic model was obtained using the Lyapunov function. Numerical simulations were carried out to validate theoretical results on extinction and persistence of COVID-19 within the population. We investigated the situations when the stochastic associated reproduction number is below one and also when it is greater than one. The associated stochastic reproduction number is given as $R_0^S = 0.2585 < 1$ which shows that the disease will go into extinction exponentially with unit probability. This is presented in figure 2. It was also observed that the stochastic system fluctuates around the endemic equilibrium. When $R_0^S = 2.4423 > 1$, the disease persists within the population, but figure 3 shows that there is a possibility for the disease to go into extinction.

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