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Bivariate Nadarajah-Haghighi distribution derived from copula functions: Bayesian estimation and applications

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Abstract. The Nadarajah-Haghighi distribution is an important lifetime distribution in survival analysis that serves as an alternative to the Weibull, gamma and exponentiated exponential distributions. In this paper, a new bivariate distribution is introduced using the Nadarajah-Haghighi distribution. The joint probability density function was obtained using two copula functions: Gumbel-Barnett and Clayton copula functions. The model was implemented under a Bayesian method of estimation, where Markov Chain Monte Carlo (*MCMC*) simulations technique was employed to estimate the parameters of the model. Applications to real data sets to show the utility of the model was provided using kidney data and diabetic retinopathy data sets. The results of the applications suggest that the new bivariate distribution fit the real data and perform much better than its competitors.

Keywords: Nadarajah-Haghighi Distribution, Bivariate Models, Copula Function, Bayesian method, Censoring and Kidney data

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1. Introduction

Bivariate data are present in areas such as engineering and medical sciences. For instance, in the medical area, one may be interested in studying the lifetimes of paired human organs, such as kidneys, eyes, double recurrence of a certain disease, familial association between various genetic diseases like breast cancer, diabetes and heart diseases and times to primary and secondary complications of a disease. Let T_1 and T_2 be the lifetimes associated to the same individual/device. In most bivariate lifetime data, the lifetime of one component may influence the lifetime of the other component. That is, the bivariate lifetimes data set presents dependence between the two lifetimes T_1 and T_2.
To study the structure of this dependence, Vaupel *et al.* (1979) have proposed

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the use of frailty models. In these models, the dependence between the lifetimes is modelled by the inclusion of one or more random effects, while the marginal times are conditionally independent given the frailty variable(s).

The use of copula functions is an alternative in modelling the dependence between the lifetimes T_1 and T_2 (see Nelsen (2007), Trivedi and Zimmer (2007), Balakrishnan and Lai (2009) and Joe (2014)). Hence, Copulas are functions that connect univariate distributions together so as to form multivariate distributions. Different types of copulas have been developed and studied by different researchers Nelsen (2007), Trivedi and Zimmer (2007), etc. Hence, different copula functions give different dependence structure among variables. Copula functions have been applied in different fields such as: medical sciences by Viswanathan and Manatunga (2001), Achcar *et al.* (2016), management sciences by Abbas (2006), finance and economics by Roch and Alegre (2006), Patton (2006) and Rivieccio (2015), (see Peres *et al.* (2018)) for more details.

The exponential distribution is a continuous distribution that is usually used in measuring the amount of time for some specific event(s) to occur. The distribution is a one parameter distribution that is well known due to the constant hazard rate, memory less property and a decreasing probability density function it possesses. Hence, choosing the exponential distribution in reliability studies may be inappropriate since its hazard rate does not show monotone and/ or non-monotone failure rate behaviours, (Tahir *et al.,* 2018). To solve this problem, researchers have generalized the exponential distribution in order to add flexibility to the distribution. For instance, Gupta and Kundu (1999) generalizes the exponential distribution to the exponentiated exponential distribution, Nadarajah and Haghighi (2006) to the Beta-exponential distribution, Nadarajah and Haghighi (2011) to the Nadarajah-Haghighi distribution. Other distributions that generalized the exponential distribution include the Weibull, Gamma, Burr X, Burr XII, double exponential distributions to mentioned but a few. The present paper, introduced bivariate distribution that could effectively modeled bivariate survival data in different situations including censored data where two lifetimes are observed for the same individual. The Nadarajah-Haghighi exponential distribution introduced by Nadarajah and Haghighi (2011) was considered since it is more flexible compared to the exponential distribution.

The rest of the paper is organized as follows: In section 2, the Nadarajah-Haghighi exponential distribution and copula functions were discussed. The survival function and pdf of the bivariate Nadarajah-Haghighi distribution were also discussed. Parameters of the distribution were estimated using the Bayesian method of estimation in section 3 while section 4 gives real life applications of the proposed methodology. Finally, we conclude in section 5.

2. Materials and Method

In this section, we give a brief discussion on Nadarajah-Haghighi exponential distribution and copula functions. The bivariate Nadarajah-Haghighi distribution were introduced using the Nadarajah-Haghighi exponential distribution considering the Gumbel-Barnett and Clayton copula functions.

2.1 *Nadarajah-Haghighi distribution*

Let *T* be a random variable denoting the time to the occurrence of an event of interest. The survival function of the Nadarajah-Haghighi exponential distribution with parameters α and β is given as:

$$S\left(t/\alpha,\beta\right)=exp\left(1-\left(1+\alpha t\right)^{\beta}\right) \tag{1}$$

where $t>0,\alpha>0$ is the scale parameter, $\beta>0$ is the shape parameter. The corresponding probability density function *(pdf)*, cumulative distribution function *(cdf)* and hazard rate function of the Nadarajah-Haghighi exponential *(NH)* distribution are respectively given as:

$$f\left(t/\alpha,\beta\right)=\alpha\beta\left(1+\alpha t\right)^{\beta-1}exp\left(1-\left(1+\alpha t\right)^{\beta}\right) \tag{2}$$

$$F\left(t/\alpha,\beta\right)=1-exp\left(1-\left(1+\alpha t\right)^{\beta}\right) \tag{3}$$

and

$$h\left(t/\alpha,\beta\right)=\alpha\beta\left(1+\alpha t\right)^{\beta-1} \tag{4}$$

The survival function in (1) reduces to the survival function of the exponential distribution when the shape parameter takes the value one(1). The shape of the *NH* density can be monotonically decreasing, while that of the hazard rate function can be increasing, decreasing or constant. The distribution was shown to be an alternative distribution to the Weibull, gamma and exponentiated exponential distributions.

2.2 *Copula*

The term copula is a Latin word which means to connect and is used to refer to connecting words. The word was first introduced by Sklar (1959) even though Hoeffding had already presented the idea and established the best possible bond for these functions in Hoeffding (1940) and Hoeffding (1941), (see Trivedi and Zimmer (2007) for more details). Hence, copulas are functions that connect multivariate distributions to their one dimensional margins. Let G be a p-dimensional cumulative distribution function with one-dimensional margins $G_1, G_2, \cdots, G_p$ then $\exists$ a p-dimensional copula C such that
$G\left(x_1, x_2, \cdots, x_p\right)=C\left(G_1\left(x_1\right), G_2\left(x_2\right), \cdots, G_p\left(x_p\right)\right)$.
The copula approach is a useful technique used in deriving the joint distribution function of a random variable given their marginal distributions especially when the variables are non-normal. The case $p=2$ was considered in this work.
The theory of copulas basically relies on Sklars theorem. The theorem states that: For any random variables, $X_1,\ X_2,\ \cdots,\ X_p$ with joint *cdf*,
$F\left(X_1,\ X_2,\ \cdots,\ X_p\right) = P\left(X_1 \leqslant x_1,\ X_2 \leqslant x_2,\ \cdots,\ X_p \leqslant x_p\right)$
and marginal $cdfs\ F_j\left(x\right) = P\left(X_j < x\right)$, for $j = 1,\ 2,\ \cdots,\ p$ then $\exists$ a copula C such that $F\left(X_1,\ X_2,\ \cdots,\ X_p\right) = C\left(F\left(x_1\right),\ F_2\left(x_2\right),\ \cdots,\ F_p\left(x_p\right)\right)$. C is unique if $F_j\left(x\right)$ is continuous.

Several types of copulas have been developed and studied. Nelsen (2007) and Trivedi and Zimmer (2007) provided a very thorough coverage of the various types of copulas. However, this study will use the Gumbel-Barnett (*GB*) and Clayton copulas in introducing bivariate *NH* distribution.

2.2.1 The Model based on Gumbel-Barnett Copula

Copula functions are used in connecting the joint distribution function of two or more univariate distributions. The copula function is said to be bivariate when it connects the joint distribution function of only two univariate distributions. Let $S(t_k)$ be the univariate survival function for the random variable $T_p, \quad p = 1, 2$ the joint survival function $S(t_1, t_2)$ is defined as:

$$S(t_1, t_2) = C_\psi (S(t_1) S(t_2)) \tag{5}$$

where $t_1 > 0$ and $t_2 > 0$, ψ is a measure of dependence between the random variables T_1 and T_2 while C is a copula function. The Gumbel-Barnett copula (GB) was proposed by Gumbel (1960), and Barnett (1980). The joint survival function considering the *GB* copula function for the random variables T_1 and T_2 is given as:

$$S(t_1, t_2) = S(t_1) S(t_2) exp(-\psi \ell n(S(t_1)) \ell n(S(t_2))) \tag{6}$$

where $\psi \in (0, 1)$ is the dependence parameter. It is important to note that, the *GB* copula covers a region of negative dependence. The dependent parameter ψ is related to the kendall (τ) coefficient as:

$$\tau(\psi) = e^{\frac{2}{\psi}} \int_{-\infty}^{-\frac{2}{\psi}} \frac{e^t}{t} dt$$

Assume T_1 and T_2 be two lifetimes associated to the same individual with a dependence structure given by *GB* copula function. Assume further, $T_1 \sim NH(\alpha_1, \beta_1)$ and $T_2 \sim NH(\alpha_2, \beta_2)$. Then, the survival functions of the marginal distributions for the lifetimes T_1 and T_2 are given by:

$$S_1(t_1) = exp\left(1 - (1 + \alpha_1 t_1)^{\beta_1}\right) \tag{7}$$

and

$$S_2(t_2) = exp\left(1 - (1 + \alpha_2 t_2)^{\beta_2}\right) \tag{8}$$

respectively, while the probability density functions are given by:

$$f_1(t_1) = \alpha_1 \beta_1 (1 + \alpha_1 t_1)^{\beta_1 - 1} exp\left(1 - (1 + \alpha_1 t_1)^{\beta_1}\right) \tag{9}$$

and

$$f_2(t_2) = \alpha_2 \beta_2 (1 + \alpha_2 t_2)^{\beta_2 - 1} exp\left(1 - (1 + \alpha_2 t_2)^{\beta_2}\right) \tag{10}$$

respectively. Hence, the joint long-term survival function for the lifetimes T_1 and T_2 considering the Gumbel-Barnett copula is given as:

$$S(t_1, t_2) = exp\left(2 - \psi\left(1 - c_1\right)\left(1 - c_2\right) - c_1 - c_2\right) \tag{11}$$

where ψ is the dependent parameter and it takes values within the interval $(0,\ 1)$, $c_1 = (1 + \alpha_1 t_1)^{\beta_1}$, $c_2 = (1 + \alpha_2 t_2)^{\beta_2}$, $\alpha_1 > 0$ and $\alpha_2 > 0$ are the scale parameters, $\beta_1 > 0$ and $\beta_2 > 0$ are the shape parameters. The lifetimes T_1 and T_2 in expression (11) becomes independent when the dependent parameter ψ takes the value zero.
The first partial derivatives of (11) with respect to t_1 and t_2 are respectively:

$$\frac{\partial S(t_1, t_2)}{\partial t_1} = \alpha_1 \beta_1 A B_1 A(t_1, t_2)\left[1 - \psi\left(1 - c_2\right)\right] \tag{12}$$

and

$$\frac{\partial S(t_1, t_2)}{\partial t_2} = \alpha_2 \beta_2 A B_2 A(t_1, t_2)\left[1 - \psi\left(1 - c_1\right)\right] \tag{13}$$

where $A = exp\left(2 - c_1 - c_2\right),\ A(t_1, t_2) = exp\left(-\psi(1 - c_1)(1 - c_2)\right),$
$B_1 = (1 + \alpha_1 t_1)^{\beta_1 - 1}$ and $B_2 = (1 + \alpha_2 t_2)^{\beta_2 - 1}$.
The joint density function for the random variables T_1 and T_2 could be obtained by deriving the second derivative of $S\left(t_1, t_2\right)$ with respect to t_1 and t_2. That is, $f\left(t_1, t_2\right) = \frac{\partial^2 S(t_1, t_2)}{\partial t_1 \partial t_2}$, this yields:

$$f(t_1, t_2) = \alpha_1 \alpha_2 \beta_1 \beta_2 B_1 B_2 S_{GB}(t_1, t_2)\left[(1 - \psi)^2 - \psi - \psi(\psi - 1)(c_1 + c_2) + \psi^2 c_1 c_2\right] \tag{14}$$

The *cdf* corresponding to (11) is given as:

$$F(t_1, t_2) = exp\left(2 - \psi(1 - c_1)(1 - c_2) - c_1 - c_2\right) \tag{15}$$

2.2.2 The Model based on Clayton Copula

The Clayton copula is popularly used in fitting bivariate lifetimes data because of its ability to describe positive dependence structure. Clayton copula function was first introduced by Clayton (1978) and was later studied by Cook (1981) and Oakes (1982). The joint survival function for the lifetimes T_1 and T_2 based on the Clayton copula function is defined as:

$$S\left(t_1,\ t_2\right) = \left((S(t_1))^{-\psi} + (S(t_2))^{-\psi} - 1\right)^{-\frac{1}{\psi}} \tag{16}$$

where $S(t_1)$ and $S(t_2)$ are the respective marginal survival functions for the random lifetimes T_1 and T_2, ψ is the dependence parameter and it takes values in the interval $(0, \infty)$. It is important to note that the random lifetimes T_1 and T_2 become independent when the dependence parameter approaches zero. The

dependence parameter ψ is related to the Kendall's (τ) coefficient as:

$$\tau(\psi) = \frac{\psi}{\psi + 2}$$

where $0 < \tau(\psi) \leq 1$. Note that total dependence between T_1 and T_2 will be observed when ψ tends to infinity. Hence, Clayton copula is adequate in modelling positive dependences and it has the advantage of measuring a high range of positive correlations.
Consider the Clayton copula function assume $T_1 \backsim NH(\alpha_1, \beta_1)$ and $T_1 \backsim NH(\alpha_2, \beta_2)$, then the joint survival function for the bivariate lifetimes T_1 and T_2 using equation (16) is given as:

$$S\left(t_1,\, t_2\right) = \left(e^{-\psi(1-c_1)} + e^{-\psi(1-c_2)} - 1\right)^{-\frac{1}{\psi}} \tag{17}$$

where $0 < \psi < \infty$ and ψ is the dependence parameter.
Theorem 1: The joint survival function for T_1 and T_2 in equation (17) reduced to $S\left(t_1,\, t_2\right) = A$ when $\psi \to 0$. That is, T_1 and T_2 are independent when $\psi \to 0$.
proof:

$$\lim_{\psi \to 0} \left(e^{-\psi(1-c_1)} + e^{-\psi(1-c_2)} - 1\right)^{-\frac{1}{\psi}}$$

let

$$u = e^{(1-c_1)} \quad and \quad v = e^{(1-c_2)}$$

then

$$\begin{aligned}
&\lim_{\psi \to 0} \left(e^{-\psi(1-c_1)} + e^{-\psi(1-c_2)} - 1\right)^{-\frac{1}{\psi}} \\
&= uv \lim_{\psi \to 0} \left(u^{\psi} + v^{\psi} - u^{\psi}v^{\psi}\right)^{-\frac{1}{\psi}} \\
&= uv \lim_{\psi \to 0} exp\left\{ -\frac{log\left(\left(u^{\psi} + v^{\psi} - u^{\psi}v^{\psi}\right)^{-\frac{1}{\psi}}\right)}{\psi} \right\}
\end{aligned}$$

applying L'hospital rule gives:

$$\begin{aligned} & uv \lim_{\psi \to 0} exp\left\{-\frac{u^{\psi}log(u)+v^{\psi}log(v)-u^{\psi}v^{\psi}log(u)-u^{\psi}v^{\psi}log(v)}{u^{\psi}+v^{\psi}-u^{\psi}v^{\psi}}\right\} \\ &= uv \lim_{\psi \to 0} exp\{0\} \\ &= uv \\ &= e^{(1-c_1)}e^{(1-c_2)} \\ &= A \end{aligned}$$

The first partial derivatives of (17) with respect to t_1 and t_2 are respectively given by:

$$\frac{\partial S(t_1,t_2)}{\partial t_1} = \alpha_1\beta_1 B_1 e^{-\psi(1-c_1)} B(t_1,t_2)^{\frac{-1}{\psi}-1} \tag{18}$$

and

$$\frac{\partial S(t_1,t_2)}{\partial t_2} = \alpha_2\beta_2 B_2 e^{-\psi(1-c_2)} B(t_1,t_2)^{\frac{-1}{\psi}-1} \tag{19}$$

where

$B(t_1,t_2) = exp\left(-\psi(1-(1+\alpha_1 t_1)_1^{\beta})\right) + exp\left(-\psi(1-(1+\alpha_2 t_2)_2^{\beta})\right) - 1.$

The *pdf* and *cdf* corresponding to (17) are respectively:

$$f(t_1\ t_2) = \alpha_1\alpha_2\beta_1\beta_2 B_1 B_2 A^{-\psi} B(t_1,t_2)^{\frac{-1}{\psi}-2} \tag{20}$$

$$F(t_1\ t_2) = B(t_1,t_2)^{\frac{-1}{\psi}} - e^{1-c_1} - e^{1-c_2} + 1 \tag{21}$$

2.3 *Inference Methods*

In this section, the problem of estimating the parameters of the bivariate Nadarajah-Haghighi distribution based on random samples of size n was addressed using the Bayesian method of estimation procedure.

2.3.1 Bayesian Method of Estimation

The Bayesian method of estimation is a method that combines the prior information with new information that is available to form the basis for statistical analysis. That is, Bayesian method combines prior information with new information to come up with the posterior distribution. To find the estimates of the model using Bayesian method, let $\Theta = (\alpha_1, \alpha_2, \beta_1, \beta_2, \psi)'$ be the vector of unknown parameters. Under the Bayesian framework, the joint posterior distribution of the parameters in the model is obtained by combining the likelihood function and the joint prior distribution of the parameters. The likelihood function of the parameters assuming right censoring is obtain as follows:
Let T_1 and T_2 be two lifetimes associated with the same subject and assume either T_1 or T_2 may be censored. Assume further that, censoring is in-

dependent of the time to the event of interest in the study. Let (T_{11}, T_{21}), $(T_{12}, T_{22}), \cdots, (T_{1n}, T_{2n})$ be a random sample from the bivariate Nadarajah-Haghighi distribution with parameter Θ where $\Theta = (\alpha_1, \alpha_2, \beta_1, \beta_2, \psi)'$ is a parameter space. Then, the ith observation $i = 1, 2, \cdots, n$ fall in one of the following groups:

- G_1 : both t_{1i} and t_{2i} are complete observations.
- G_2 : t_{1i} is complete and t_{2i} is censored.
- G_3 : t_{1i} is censored and t_{2i} is not censored.
- G_4 : both t_{1i} and t_{2i} are censored observations.

Then, the likelihood based on these conditions can be expressed as:

$$L = \prod_{i \in G_1} \left[\frac{\partial^2 S\left(t_{1i}, t_{2i}\right)}{\partial t_{1i} \partial t_{2i}} \right] \prod_{i \in G_2} \left[\frac{-\partial S\left(t_{1i}, t_{2i}\right)}{\partial t_{1i}} \right] \prod_{i \in G_3} \left[\frac{\partial S\left(t_{1i}, t_{2i}\right)}{\partial t_{2i}} \right] \prod_{i \in G_4} S\left(t_{1i}, t_{2i}\right) \tag{22}$$

Let δ_{1i} and δ_{2i} be indicator variables, such that $\delta_{1i} = \delta_{2i} = 1$ when t_{ji} is a complete observation and $\delta_{1i} = \delta_{2i} = 0$ when t_{ji} is censored. Hence, the likelihood function in equation (22) can now be written as:

$$L = \prod_{i=1}^{n} \left[\frac{\partial^2 S\left(t_{1i}, t_{2i}\right)}{\partial t_{1i} \partial t_{2i}} \right]^{\delta_{1i}\delta_{2i}} \left[\frac{-\partial S\left(t_{1i}, t_{2i}\right)}{\partial t_{1i}} \right]^{\delta_{1i}(1-\delta_{2i})} \left[\frac{-\partial S\left(t_{1i}, t_{2i}\right)}{\partial t_{2i}} \right]^{(1-\delta_{1i})\delta_{2i}} \left[S\left(t_{1i}, t_{2i}\right) \right]^{(1-\delta_{1i})(1-\delta_{2i})} \tag{23}$$

Let the joint prior distribution for $\alpha_1, \alpha_2, \beta_1, \beta_2$ and ψ be

$$\Pi(\Theta) = \pi_{11}(\alpha_1)\pi_{12}(\alpha_2)\pi_{21}(\beta_1)\pi_{22}(\beta_2)\pi_3(\psi)$$

Assumed the following prior distributions for the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ and ψ of the ***BNH*** distribution:

$$\pi_{1k}(\alpha_k) \propto \alpha_k^{b_k - 1} e^{a_k \alpha_k} \qquad \alpha_k > 0$$

and

$$\pi_{2k}(\beta_k) \propto \beta_k^{d_k - 1} e^{c_k \beta_k} \qquad \beta_k > 0$$

for $k = 1, 2$. All the hyper-parameters a_k, b_k, c_k and d_k are assumed to be known and non-negative. On the other hand, a uniform prior was assumed for the dependence parameter (ψ) when considering the ***GB*** copula function. That is:

$$\pi_3(\psi) \sim unif\left[e, f\right]$$

where the hyper-parameters e and f are assumed to be known and real. While a gamma prior was assumed for the dependence parameter (ψ) when considering the Clayton model. That is:

$$\pi_3(\psi) \propto \psi^{h-1} e^{g\psi} \qquad \psi > 0 \tag{24}$$

where g and h are known hyper-parameters. The hyper-parameters g and h were also assumed to be non-negative. Furthermore, we assumed independence be-

tween the prior distributions of the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ and ψ. Then, considering as special case, the *BNH* distribution based on the *GB* copula, suppose that $x_1, x_2, \cdots, x_n$ is a random sample from the ***BNH-GB*** $\sim (\alpha_1, \alpha_2, \beta_1, \beta_2, \psi)$, then, using the likelihood function together with $\Pi(\Theta)$ (the joint prior distribution), the joint posterior density for $\alpha_1, \alpha_2, \beta_1, \beta_2$ and ψ could be written as:

$$\ell(\alpha_1, \alpha_2, \beta_1, \beta_2, \psi / x_1, x_2, \cdots, x_n) = \frac{\ell(x_1, x_2, \cdots, x_n / \alpha_1, \alpha_2, \beta_1, \beta_2, \psi)\Pi(\Theta)}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \ell(x_1, x_2, \cdots, x_n / \alpha_1, \alpha_2, \beta_1, \beta_2, \psi)\Pi(\Theta) d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 d\psi} \tag{25}$$

Computing the estimates of the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ and ψ analytically in (25) may not be possible in this case. Hence, ***MCMC*** method will be employed to generate the samples from the posterior distribution and then compute the Bayesian estimates of the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ and ψ. The OpenBug software was used in this work to obtain the Bayes estimates.

2.4 *Deviance Information Criteria*

The Deviance Information Criteria was used in assessing the performance of the fits of two or more models. The Deviance Information Criteria (DIC) proposed by Spiegelhalter *et al.* (2002) is defined as $D(\Theta) = D\left(\hat{\Theta}\right) + 2n_p = 2\bar{D} - D\left(\hat{\Theta}\right)$, where $D\left(\hat{\Theta}\right)$ is the deviance evaluated at the posterior mean $\hat{\Theta}$, n_p is the effective number of parameters in the model defined as $n_p = \bar{D} - D\left(\hat{\Theta}\right)$, $\bar{D} = E(D(\Theta))$ is the posterior deviance measuring the goodness of fit of the model to a given data set. The model with the least ***DIC*** value is regarded as the best model.

3. Results and Discussion

In this section, two real data sets were analyzed in order to demonstrate the applicability of the proposed models. The first data set is the infections in kidney patients data from McGilchrist and Aisbett (1991) which was previously analyzed by Achcar *et al.* (2015), Elaal and Jarwan (2017) and Mirhosseini *et al.* (2015). The recurrence times to infection at point of insertion of catheter using portable dialysis equipment for thirty-eight (38) kidney patients were recorded. Two recurrence times were recorded for each patient together with censoring indicator (Infection occurs =1 and censored=0) and risk variable values (age, sex: male=1, female=2 and disease type). Let T_1 and T_2 refers to first and second recurrence time respectively. The recurrence times together with the aforementioned variables are shown in Table 1.

Table 1: Kidney data

Patient	T_1	T_2	Event type 1	Event type 2	Sex	Age	Disease types
1	8	16	1	1	1	28	3
2	23	13	1	0	0	48	0
3	22	28	1	1	1	32	3
4	447	318	1	1	0	31.5	3
5	30	12	1	1	1	10	3
6	24	245	1	1	0	16.5	3
7	7	9	1	1	1	51	0
8	511	30	1	1	0	55.5	0
9	53	196	1	1	0	69	1
10	15	154	1	1	1	51.5	0
11	7	333	1	1	0	44	1
12	141	8	1	0	0	34	3
13	96	38	1	1	0	35	1
14	149	70	0	0	0	42	1
15	536	25	1	0	0	17	3
16	17	4	1	0	1	60	1
17	185	177	1	1	0	60	3
18	292	114	1	1	0	43.5	3
19	22	159	0	0	0	53	0
20	15	108	1	0	0	44	3
21	152	562	1	1	1	46.5	2
22	402	24	1	0	0	30	3
23	13	66	1	1	0	62.5	1
24	39	46	1	0	0	42.5	1
25	12	40	1	1	1	43	1
26	113	201	0	1	0	57.5	1
27	132	156	1	1	0	10	0
28	34	30	1	1	0	52	1
29	2	25	1	1	1	53	0
30	130	26	1	1	0	54	0
31	27	58	1	1	0	56	1
32	5	43	0	1	0	50.5	1
33	152	30	1	1	0	57	2
34	190	5	1	0	0	44.5	0
35	119	8	1	1	0	22	3
36	54	16	0	0	0	42	3
37	6	78	0	1	0	52	2
38	63	8	1	0	1	60	2

The data was fitted to the bivariate Nadarajah-Haghighi distributions considering Gumbel-Barnett and Clayton copula function and compared their performance with the fits of bivariate exponential (*BE*), bivariate Weibull (*BW*), bivariate exponentiated exponential (*BEE*) and bivariate modified Weibull (*BMW*) distributions.

Table 2: Posterior Summary Statistics Using Gumbel-Barnett Copula Function - Kidney Data

model	parameter	mean	sd	95% *CrI*	DIC
BNH	ψ	0.1261	0.1150	(0.0037, 0.4189)	684.6
	α_1	0.4700	0.1493	(0.2645, 0.8489)	
	α_2	0.6980	0.2699	(0.3413, 1.3790)	
	β_1	0.04371	0.0313	(0.0096, 0.1267)	
	β_2	0.01891	0.0135	(0.0043, 0.0551)	
BE	ψ	0.0978	0.0887	(0.0027, 0.3330)	689.1
	α_1	0.0076	0.0013	(0.0052, 0.0105)	
	α_2	0.0076	0.0015	(0.0050, 0.0108)	
BW	α_1	0.7436	0.0993	(0.5601, 0.9545)	688.7
	α_2	0.8991	0.1313	(0.6567, 1.1790)	
	ψ	0.1203	0.1114	(0.0040, 0.4151)	
	β_1	0.0327	0.0174	(0.0090, 0.0756)	
	β_2	0.0155	0.0110	(0.003, 0.0444)	
BEE	α_1	0.7527	0.1585	(0.4789, 1.0950)	691.2
	α_2	1.0420	0.2457	(0.6393, 1.5850)	
	ψ	0.1091	0.1041	(0.0031, 0.3895)	
	β_1	0.0061	0.0016	(0.0034, 0.0095)	
	β_2	0.0077	0.0020	(0.0042, 0.0121)	
BMW	α_1	0.0457	0.0272	(0.0103, 0.1169)	690.8
	α_2	0.0265	0.0174	(0.0062, 0.0702)	
	β_1	0.637	0.1346	(0.3841, 0.9064)	
	β_2	0.7275	0.1368	(0.4628, 0.9989)	
	ϕ_1	0.0010	0.0007	(0.0000, 0.0026)	
	ϕ_2	0.0013	0.0008	(0.0001, 0.0032)	
	ψ	0.0829	0.0701	(0.0027, 0.2629)	

In analyzing this data, we assumed prior independence among the model parameters and consider the following prior densities for $\alpha_k,\ \beta_k$ and ψ: $\alpha_k \sim Gamma(1,1),\ \beta_k \sim Gamma(1,1)$ where $k = 1, 2$, while $\psi \sim Unif(0,1)$ when considering the Gumbel-Barnett copula and $\psi \sim Gamma(1,1)$ when considering the Clayton copula. Bayesian summary statistics were obtained by using Markov Chain Monte Carlo (*MCMC*) simulation. In the applications, 220,000 Gibbs samples for each model parameter were generated and the first 20,000 simulated samples were discarded as burn-in so as to minimize the effect of initial values. Also, to avoid auto-correlation between successive samples, each $20th$ simulated sample was stored. Hence, inferences were based on 10,000 samples.

Table 2 gives the results of the fits of the *BNH*, *BE*, *BW*, *BEE* and *BMW* distributions considering the Gumbel-Barnett (*GB*) copula. The table gives the posterior mean estimates of the parameters together with the standard deviation of the estimates, 95% credible interval(95% *CrI*) and the *DIC* values. The results showed that, the *BNH* distribution has the least *DIC* value. Hence, it is more efficient compared to other models considered in the work.

Table 3: Posterior Summary Statistics Using Clayton Copula Function - Kidney Data

model	parameter	mean	sd	95% *CrI*	DIC
BNH	ψ	0.0580	0.0583	(0.0014, 0.2166)	685.8
	α_1	0.4816	0.1431	(0.2756, 0.8341)	
	α_2	0.6551	0.2540	(0.3213, 1.3110)	
	β_1	0.0419	0.0282	(0.0099, 0.1167)	
	β_2	0.0231	0.0166	(0.0049, 0.0667)	
BE	ψ	0.3265	0.2253	(0.0197, 0.8593)	687.1
	α_1	0.0077	0.0013	(0.0053, 0.0105)	
	α_2	0.0075	0.0014	(0.0049, 0.0105)	
BW	α_1	0.7424	0.0984	(0.559, 0.9495)	687.2
	α_2	0.8739	0.1240	(0.6395, 1.125)	
	ψ	0.4463	0.3067	(0.0284, 1.1800)	
	β_1	0.0331	0.0176	(0.0096, 0.07700)	
	β_2	0.0169	0.0113	(0.0036, 0.0464)	
BEE	α_1	0.7517	0.1560	(0.4838, 1.1000)	688.6
	α_2	1.0500	0.2432	(0.6501, 1.6010)	
	ψ	0.3498	0.2495	(0.0211, 0.9744)	
	β_1	0.0062	0.0015	(0.0035, 0.0094)	
	β_2	0.0077	0.0019	(0.0043, 0.0119)	
BMW	α_1	0.4421	0.2081	(0.0598, 0.7879)	689.9
	α_2	0.3659	0.1455	(0.0499, 0.6502)	
	β_1	0.2478	0.1232	(0.0002, 0.4235)	
	β_2	0.2020	0.0942	(0.0010, 0.3351)	
	ϕ_1	0.0018	0.0007	(0.0005, 0.0035)	
	ϕ_2	0.0019	0.0007	(0.0007, 0.0034)	
	ψ	0.1934	0.0689	(0.1000, 0.2730)	

Table 3 gives the summary statistics of the results of the fits of the aforementioned distributions considering the Clayton copula function. Also, the *BNH* distribution fits the data better than the *BE*, *BW*, *BEE* and *BMW* distributions. Furthermore, comparing between the fits of the bivariate models considering the *GB* and the Clayton models showed that, overall, the *BNH* distribution considering the *GB* copula function best fits the data.

The second data set is the diabetic retinopathy data set given by Huster *et al.* (1989), available in the R package *SurvCor* Ploner *et al.* (2015). This data was analyzed by Achcar *et al.* (2016), Franco *et al.* (2020), Martinez *et al.* (2018), Peres *et al.* (2018). The data consists of the follow-up times of 197 diabetic patients under the age of 60 years. The study aimed to assess the efficacy of photocoagulation treatment for proliferative retinopathy. Each eye of each patient was randomized to either laser treatment or no treatment (being used as the control). The event of interest is severe visual loss in each eye of the patient. Let T_1 and T_2 be respectively the time up to visual loss for the control eye and the treatment eye. Censoring was caused by death, dropout or termination of the study.

In analyzing this data set, the choice of prior distribution of the model parameters was the same with that of the kidney data application. Table 4 showed re-

Table 4: Posterior Summary Statistics Using Gumbel-Barnett Copula Function - Diabetic Retinopathy

model	parameter	mean	sd	95% *CrI*	DIC
BNHD	ψ	0.0561	0.053	(0.0016, 0.1940)	1692
	α_1	0.2586	0.0876	(0.1482, 0.4955)	
	α_2	0.3135	0.0841	(0.1941, 0.5281)	
	β_1	0.0858	0.0464	(0.0238, 0.2014)	
	β_1	0.0852	0.0394	(0.0305, 0.1820)	
BE	ψ	0.0484	0.0435	(0.0014, 0.1615)	1711
	α_1	0.0096	0.0011	(0.0075, 0.0119)	
	α_2	0.0129	0.0014	(0.0103, 0.0157)	
BW	α_1	0.7612	0.0840	(0.6016, 0.9297)	1704
	α_2	0.7876	0.0745	(0.6484, 0.9391)	
	ψ	0.0539	0.0518	(0.0014, 0.1872)	
	β_1	0.0248	0.0084	(0.0119, 0.0444)	
	β_1	0.0294	0.0087	(0.0156, 0.0497)	
BEE	α_1	0.7800	0.1032	(0.5914, 0.9989)	1736
	α_2	0.7949	0.0967	(0.6139, 1.0000)	
	ψ	0.0524	0.0489	(0.0016, 0.1805)	
	β_1	0.0067	0.0017	(0.0037, 0.0103)	
	β_1	0.0095	0.0020	(0.0059, 0.0137)	
BMW	α_1	0.0268	0.0087	(0.0129, 0.0471)	1706
	α_2	0.0304	0.0093	(0.0154, 0.0518)	
	β_1	0.7054	0.0905	(0.5273, 0.8816)	
	β_1	0.7564	0.0823	(0.5986, 0.9204)	
	ϕ_1	0.0027	0.0023	(0.0001, 0.0085)	
	ϕ_2	0.0019	0.0017	(0.0001, 0.0065)	
	ψ	0.0509	0.0495	(0.0015, 0.1825)	

sults for the *BNH*, *BE*, *BW*, *BEE* and *BMW* distributions based on the Gumbel-Barnett copula function. The results showed that, the *BNH* distribution is more efficient compared to the *BE*, *BW*, *BEE* and *BMW* distributions since it has the least *DIC* value. On the other hand, Table 5 showed results for the *BNH*, *BE*, *BW*, *BEE* and *BMW* distributions based on the Clayton copula function considering the diabetic retinopathy data. Similar to the results of the distributions based on Gumbel-Barnett copula, Table 5 showed that, the *BNH* distribution has the lowest *DIC* value. Hence, it is more efficient compared to the *BE*, *BW*, *BEE* and *BMW* distributions. Additionally, As in the previous application, comparing between the fits of the bivariate distributions based on the Gumbel-Barnett and Clayton copula functions showed that, the *BNH* Gumbel-Barnett distribution fits the data better than the *BNH* Clayton distribution.

Table 5: Posterior Summary Statistics Using Clayton Copula Function - Diabetic Retinopathy

model	parameter	mean	sd	95% *CrI*	DIC
BNHD	ψ	0.0437	0.0434	(0.0011, 0.1602)	1693
	α_1	0.2579	0.0852	(0.148, 0.4746)	
	α_2	0.3027	0.0760	(0.1911, 0.4890)	
	β_1	0.0858	0.0450	(0.0251, 0.1983)	
	β_1	0.0930	0.0428	(0.0343, 0.1980)	
BE	ψ	0.5405	0.2232	(0.1375, 0.9976)	1703
	α_1	0.0097	0.0012	(0.0076, 0.0121)	
	α_2	0.0129	0.0014	(0.0104, 0.0159)	
BW	α_1	0.7672	0.0828	(0.6129, 0.9396)	1696
	α_2	0.7899	0.0745	(0.6517, 0.9414)	
	ψ	0.5754	0.2465	(0.1462, 1.1130)	
	β_1	0.0246	0.0082	(0.0117, 0.0435)	
	β_1	0.0294	0.0086	(0.0153, 0.0487)	
BEE	α_1	0.4715	0.358	(0.0705, 0.9666)	2007
	α_2	0.5027	0.3348	(0.129, 0.9733)	
	ψ	0.3779	0.2347	(0.1688, 0.9009)	
	β_1	0.0038	0.0036	(0.0000, 0.0099)	
	β_1	0.0054	0.005	(0.0000, 0.0133)	
BMW	α_1	0.0319	0.01	(0.0160, 0.0546)	1702
	α_2	0.0341	0.0099	(0.0181, 0.0560)	
	β_1	0.6516	0.0895	(0.4818, 0.8302)	
	β_1	0.7197	0.0801	(0.5666, 0.8789)	
	ϕ_1	0.004	0.0027	(0.0004, 0.0106)	
	ϕ_2	0.0027	0.002	(0.0002, 0.0077)	
	ψ	0.5565	0.2392	(0.145, 1.0700)	

4. Conclusion

In this paper, a new bivariate lifetime distribution based on the Nadarajah-Haghighi distribution was proposed using the Gumbel-Barnett and Clayton copula functions in the presence of right censored data. The methodology was then applied to two real data sets: kidney data and diabetic retinopathy data sets. The performance of the fits of the *BNH* distribution was compared with the fits of *BE*, *BW*, *BEE* and *BMW* distributions. *MCMC* technique was used to obtained posterior summary statistics of the fitted bivariate distributions. The fits were compared with respect to *DIC* criterion. Based on the *DIC*, the proposed *BNH* distribution considering the *GB* copula best fit the data. Finally, it is important to mention that, although the gamma prior was used for the shape and scale parameters, the method can be used for a more general class of priors. Choosing a proper prior in practice is a very difficult task. Hence, more work is needed in this direction.

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References

Abbas, A. E. (2006). Entropy methods for joint distributions in decision analysis. IEEE Transactions on Engineering Management, 53(1), 146-159.

Achcar, J. A., Martinez, E. Z., and Tovar, C. J. R. (2016). Bivariate lifetime modelling using copula functions in presence of mixture and non-mixture cure fraction models, censored data and covariates. Model Assisted Statistics and Applications, 11(4),261-276.

Achcar, J. A., Moala, F. A., Tarumoto, M. H., and Coladello, L. F. (2015). A bivariate generalized exponential distribution derived from copula functions in the presence of censored data and covariates. Pesquisa Operacional, 35, 165-186.

Balakrishnan, N. and Lai, C. D. (2009). Continuous bivariate distributions. Springer Science Business Media.

Barnett, V. (1980). Some bivariate uniform distributions. Communications in statistics-theory and methods, 9(4), 453-461.

Clayton, D. G. (1978). A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. Biometrika, 65(1), 141-151.

Cook, R. D. and Johnson, M. E. (1981). A family of distributions for modelling non-elliptically symmetric multivariate data. Journal of the Royal Statistical Society: Series B (Methodological), 43(2), 210-218.

Elaal, M. K. A. and Jarwan, R. S. (2017). Inference of bivariate generalized exponential distribution based on copula functions. Applied Mathematical Sciences, 11(24),1155-1186.

Franco, M., Vivo, J. M., and Kundu, D. (2020). A generator of bivariate distributions: Properties, estimation, and applications. Mathematics, 8(10), 1776.

Gumbel, E. J. (1960). Bivariate exponential distributions. Journal of the American Statistical Association, 55(292), 698-707.

Gupta, R. D. and Kundu, D. (1999). Theory methods: Generalized exponential distributions. Australian New Zealand Journal of Statistics, 41(2), 173-188.

Hoeffding, W. (1940). Masstabinvariante korrelationstheorie. Schriften des Mathematischen Instituts und Instituts fur Angewandte Mathematik der Universitat Berlin, 5, 181-233.

Hoeffding, W. (1941). Masstabinvariante korrelationsmasse fur diskontinuierliche verteilungen. Archiv fur mathematische Wirtschafts-und Sozialforschung, 7, 49-70.

Huster, W. J., Brookmeyer, R., and Self, S. G. (1989). Modelling paired survival data with covariates. Biometrics, 145-156.

Joe, H. (2014). Dependence modeling with copulas. CRC press.

Martinez, E. Z., Achcar, J. A., and Icuma, T. R. (2018). Bivariate basu-dhar geometric model for survival data with a cure fraction. Electronic Journal of Applied Statistical Analysis, 11(2), 655-673.

McGilchrist, C. and Aisbett, C. (1991). Regression with frailty in survival analysis. Biometrics, 461-466.