

On Rayleigh-Exponentiated Odd Generalized-Pareto Distribution with its Applications

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Abstract. In this article, we present a new *Rayleigh-Exponentiated Odd Generalized-Pareto (REOGP)* distribution. It is a member of *Rayleigh-Exponentiated Odd Generalized-X* family introduced in earlier studies. The distribution function (CDF), probability density (PDF), survival function and the hazard rate function of the new family member were provided. Furthermore, other mathematical properties such as moments, moment generating function, quantile function, entropy and function of order statistics were derived. The model parameters were estimated using the method of maximum likelihood (MLE); after which some asymptotic confidence bounds (ACB) of the parameters were obtained via utilization of the second derivatives of the likelihood function. In order to showcase the level of efficiency and consistency of *REOGP*; a Monte-Carlo simulation was conducted and it was found that, the biases, variances and the Mean Square Error (MSE) of the given parameter values approach zero as the sample size gets larger. The performance of the proposed distribution was compared (based on some real datasets) with other recent (extensions of Pareto) distributions reported in earlier studies and the results obtained indicate that, *REOGP* is suitable in modeling datasets having diverse shapes of hazard rate functions and at the same time it has proven itself to be competitive as it was able to outperform all the other competing distributions.

Keywords: T-X family, Moments, Quantile function, Maximum Likelihood, Confidence bounds.

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1. Introduction

The complex nature of problems revolving around modern day real-world phenomena; makes it quite challenging for some well-known classical probability distributions to adequately describe datasets from such phenomena. However, several researches were conducted before and many ongoing to address this

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anomaly. There are numerous studies aimed at generalizing probability models through addition of one or more parameters to an existing model. Some older researches in this regard include Pearson (1895), Burr (1942), Johnson (1949), Azzalini (1985), Mudholkar and Srivastava (1993), Marshall and Olkin (1997) as well as Gupta *et al.* (1998).

Some new ideas of generalizing probability distributions, in the early part of the 20th Century, began with the work on Beta-G family due to Eugene *et al.* (2002), Transmuted-G family due to Shaw and Buckley (2007), gamma-G family due to Zografos and Balakrishanan (2009), Kumaraswamy-G family due to Cordeiro and de Castro (2011), McDonald-G family due to Alexander *et al.*, (2012), T-X family due to Alzaatreh *et al.* (2013b), the exponentiated T-X family due to Alzaghal *et al.* (2013), the Weibull-G family due to Bourguignon *et al.* (2014), a quantile based T-XY approach due to Aljarrah *et al.* (2014), the Kumaraswamy Odd Log-logistic-G due to Alizadeh *et al.* (2016) and the logistic-G family due to Tahir *et al.* (2016).

Recently, some researches comprising Mohammed and Yahaya (2019), Yahaya and Abdullahi (2019), Eraikhuemen *et al.* (2019), Ekum *et al.* (2020) as well as Umar and Zakari (2020) are worthy of mention. Most recently, Yahaya and Doguwa (2021) proposed Rayleigh-Exponentiated Odd Generalized-X family of distributions using an exponentiated odd ratio of the distribution function of a (baseline) random variable X while utilizing the novel idea introduced in Alzaatreh *et al.* (2013b).

2. Materials and methods

2.1 The Rayleigh-Exponentiated Odd Generalized-X Family of Distributions

According to Yahaya and Doguwa (2021); a CDF of Rayleigh-Exponentiated Odd Generalized-X family of distributions having $(r+2)$ parameters is given by:

$$F(x; \alpha, \gamma, \Theta) = 1 - e^{-\frac{\gamma}{2} \left\{ \frac{G_{\Theta}^{\alpha}(x)}{\bar{G}_{\Theta}^{\alpha}(x)} \right\}^2} \quad \forall x, \alpha, \gamma > 0; \quad \Theta > 0 \quad (1)$$

while the corresponding PDF, survival and hazard functions obtainable from equation (1) are respectively given by:

$$f(x; \alpha, \gamma, \Theta) = \frac{\alpha \gamma g_{\Theta}(x) G_{\Theta}^{2\alpha-1}(x)}{[\bar{G}_{\Theta}^{\alpha}(x)]^3} e^{-\frac{\gamma}{2} \left\{ \frac{G_{\Theta}^{\alpha}(x)}{\bar{G}_{\Theta}^{\alpha}(x)} \right\}^2} \quad \forall x, \alpha, \gamma > 0; \quad \Theta > 0 \quad (2)$$

$$S(x; \alpha, \gamma, \Theta) = 1 - F(x; \alpha, \gamma, \Theta) = e^{-\frac{\gamma}{2} \left\{ \frac{G_{\Theta}^{\alpha}(x)}{\bar{G}_{\Theta}^{\alpha}(x)} \right\}^2} \quad \forall x, \alpha, \gamma > 0; \quad \Theta > 0 \quad (3)$$

$$h(x; \alpha, \gamma, \Theta) = \frac{f(x; \alpha, \gamma, \Theta)}{S(x; \alpha, \gamma, \Theta)} = \alpha \gamma g_{\Theta}(x) G_{\Theta}^{2\alpha-1}(x) [\bar{G}_{\Theta}^{\alpha}(x)]^{-3}. \quad (4)$$

where α represents the exponentiated parameter and γ is the parameter of Rayleigh distribution, $G_{\Theta}(x)$ and $g_{\Theta}(x)$ represent the CDF and PDF of the baseline random variable X indexed by an r -parameter vector $\Theta = (\theta_1, \theta_2, \dots, \theta_r)$ and $\bar{G}_{\Theta}^{\alpha}(x) = [1 - G_{\Theta}^{\alpha}(x)]$.

2.2 The Rayleigh-Exponentiated Odd Generalized-Pareto (REOGP) Distribution

If the baseline random variable X follows a Pareto distribution indexed by two parameters say, $(\beta > 0, \lambda > 0)$ whose PDF and CDF are respectively given by: $g_{\Theta}(x) = \frac{\beta\lambda^{\beta}}{x^{(\beta+1)}}$ and $G_{\Theta}(x) = 1 - \left(\frac{\lambda}{x}\right)^{\beta}$; hence, one can write Θ as $\Theta = (\beta, \lambda)$. Thus, the CDF, PDF, survival and hazard functions of *REOGP* distribution, according to equations (1) to (4) are respectively given by:

$$F(x; \alpha, \beta, \gamma, \lambda) = 1 - e^{-\frac{\gamma}{2} \left\{ \frac{\left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}}{\left(1 - \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}\right)} \right\}^2} \quad (5)$$

$$\begin{aligned} f(x; \alpha, \beta, \gamma, \lambda) &= \alpha\beta\gamma\lambda^{\beta} \left(\frac{1}{x}\right)^{\beta+1} \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{2\alpha-1} \\ &\times \left(1 - \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}\right)^{-3} e^{-\frac{\gamma}{2} \left\{ \frac{\left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}}{\left(1 - \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}\right)} \right\}^2} \end{aligned} \quad (6)$$

$$S(x; \alpha, \beta, \gamma, \lambda) = e^{-\frac{\gamma}{2} \left\{ \frac{\left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}}{\left(1 - \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}\right)} \right\}^2} \quad (7)$$

$$h(x; \alpha, \beta, \gamma, \lambda) = \alpha\beta\gamma\lambda^{\beta} \left(\frac{1}{x}\right)^{\beta+1} \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{2\alpha-1} \left(1 - \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}\right)^{-3} \quad (8)$$

where $(x > \lambda > 0; \alpha, \beta, \gamma > 0)$.

It should be noted that, *REOGP* belongs to the class of exponentiated- G family of distributions; since the CDF in (5) and the PDF in (6) can be expressed respectively as:

$$F(x; \alpha, \beta, \gamma, \lambda) = \sum_{i,j=0}^{\infty} w_{i,j} \Delta_{\alpha(2i+j+2)}(x) \quad (9)$$

$$f(x; \alpha, \beta, \gamma, \lambda) = \sum_{i,j=0}^{\infty} w_{i,j} \delta_{\alpha(2i+j+2)}(x) \quad (10)$$

where $w_{i,j} = \frac{(-1)^i \gamma^{i+1} (2i+j+1)!}{2^i i! j! (2i+2)!}$, $\Delta_k(x) = G_{\Theta}^k(x)$
and $\delta_k(x) = k g_{\Theta}(x) G_{\Theta}^{k-1}(x)$

The following figures show some possible plots of the PDF, CDF, survival as well as hazard function of *REOGP* distribution

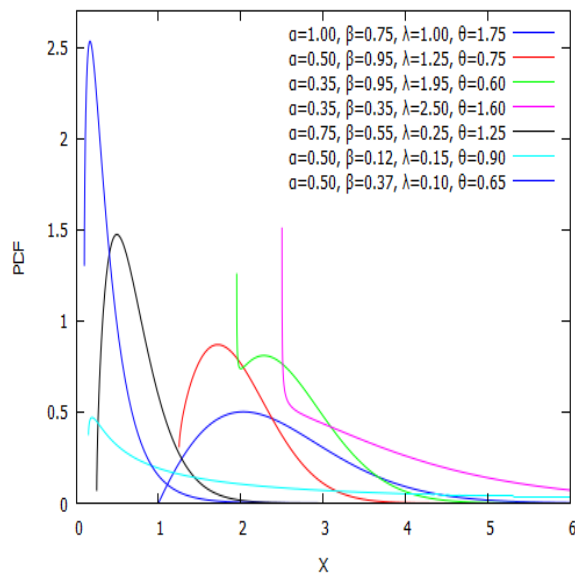


Figure 1: some *REOGP* PDF plots

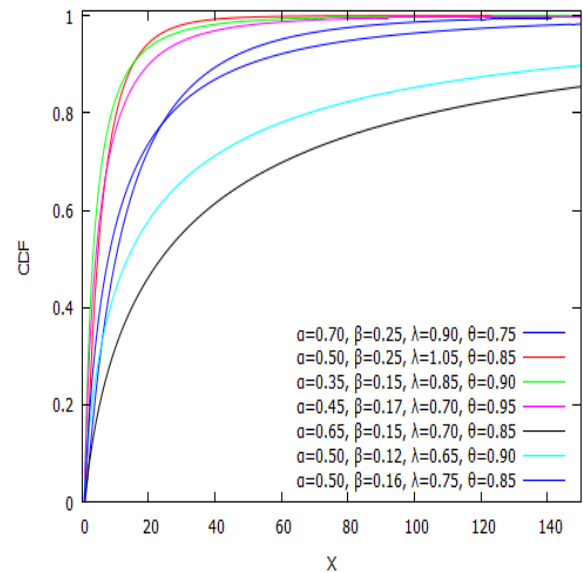


Figure 2: some *REOGP* CDF plots

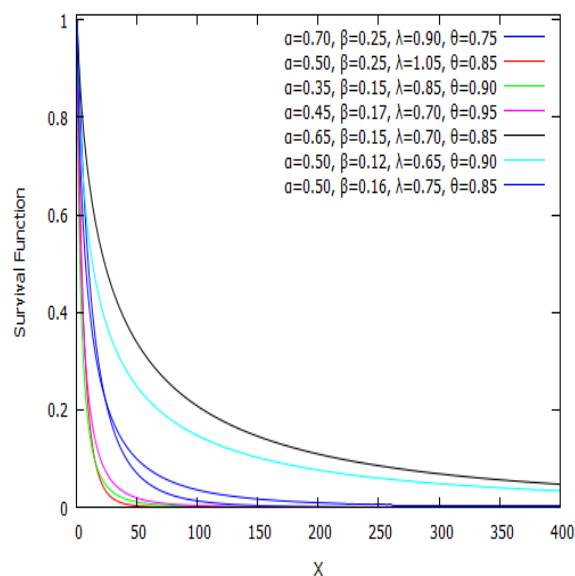


Figure 3: some *REOGP* Survival function plots

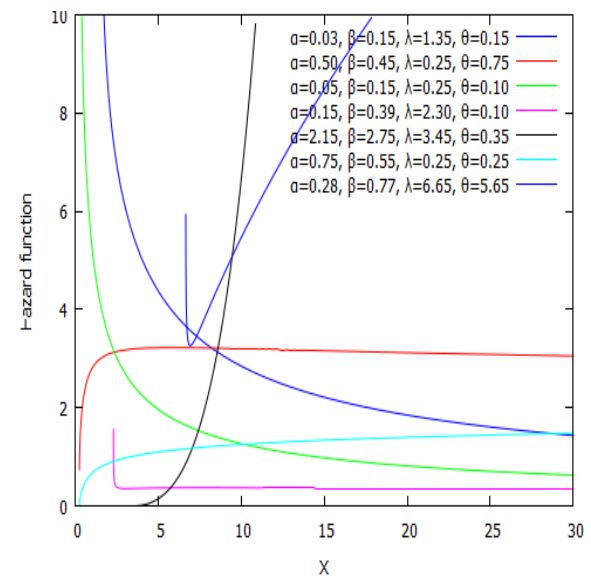


Figure 4: some *REOGP* Hazard rate function plots

Based on different shapes taken by the PDF in Figure 1 and hazard rate function in Figure 4; it can be easily observed that, *REOGP* distribution can be

used to model different types of lifetime datasets. The fact that the hazard rate function takes on varying shapes with different outlooks such as an increasing, decreasing, constant, L-shaped, inverted L-shaped, inverted J-shaped or even a V-shaped hazard rate feature is indicative of the *REOGP* distribution's capability to model different types of data sets.

2.2.1 Checking validity of *REOGP* Distribution

The validity of the PDF in equation (6) can be ascertained by showing that,

$$\int_{-\infty}^{\infty} f(x; \alpha, \beta, \gamma, \lambda) dx = 1.$$

Thus,

$$\int_{\lambda}^{\infty} f(x; \alpha, \beta, \gamma, \lambda) dx = \int_{\lambda}^{\infty} \alpha \beta \gamma \lambda^{\beta} \left(\frac{1}{x}\right)^{\beta+1} \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{2\alpha-1} \left(1 - \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}\right)^{-3} e^{-\frac{\gamma}{2} \left\{ \frac{\left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}}{\left(1 - \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}\right)} \right\}^2} dx \quad (11)$$

Now, by letting

$$u = \frac{\gamma}{2} \left\{ \frac{\left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}}{\left(1 - \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}\right)} \right\}^2 = \frac{\gamma}{2} \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{2\alpha} \left(1 - \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}\right)^{-2} \quad (12)$$

$$dx = \frac{x^{(\beta+1)} \left(1 - \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{\alpha}\right)^3 du}{\alpha \beta \gamma \lambda^{\beta} \left(1 - \left(\frac{\lambda}{x}\right)^{\beta}\right)^{2\alpha-1}} \quad (13)$$

From (12), it can be inferred that as $x \rightarrow \lambda$; $u \rightarrow 0$ and as $x \rightarrow \infty$; $u \rightarrow \infty$. Hence, after some algebraic exercises, (11) can be finally written as:

$$\int_{\lambda}^{\infty} f(x; \alpha, \beta, \gamma, \lambda) dx = \int_0^{\infty} e^{-u} du = 1 \quad (14)$$

Therefore, *REOGP* is a valid density function as required.

2.2.2 The Asymptotes of *REOGP* Distribution

The asymptotic behavior of *REOGP* distribution as the random variable X approaches (both) its lower and upper limits can be achieved by determining the limits: $\lim_{x \rightarrow \lambda} f(x; \alpha, \beta, \gamma, \lambda)$ and $\lim_{x \rightarrow \infty} f(x; \alpha, \beta, \gamma, \lambda)$. Thus

$$\lim_{x \rightarrow \lambda} f(x; \alpha, \beta, \gamma, \lambda) = \lim_{x \rightarrow \lambda} h(x; \alpha, \beta, \gamma, \lambda) \rightarrow \begin{cases} 0 & \text{if } \alpha > 1/2 \\ \frac{\beta\gamma}{2\lambda} & \text{if } \alpha = 1/2 \\ \infty & \text{if } \alpha < 1/2 \end{cases} \quad (15)$$

Also,

$$\lim_{x \rightarrow \infty} f(x; \alpha, \beta, \gamma, \lambda) = \lim_{x \rightarrow \infty} h(x; \alpha, \beta, \gamma, \lambda) \rightarrow 0 \quad (16)$$

From (15), it is obvious to the reader that, as the random variable x approaches its lower limit within its domain and given the value taken by the scale parameter α ; the PDF in (6) takes on values zero, a constant or infinity as the case may be. On the other hand, (6) approaches zero whenever the random variable approaches its maximum value within its domain irrespective of the value taken by the scale parameter α . All these facts, among other things, indicate the existence of at least a mode for *REOGP* distribution which can be obtained by solving $\frac{d}{dx} f(x; \alpha, \beta, \gamma, \lambda) = 0$.

2.3 Properties of *REOGP* Distribution

Some properties of *REOGP* distribution comprising moments, moment generating function, quantile function, Entropy measures, and distributions of Order Statistics are provided under this section.

The r^{th} non-central moment, μ'_r of *REOGP* distribution is given by:

$$E(x^r) = \mu'_r = \sum_{i,j,k=0}^{\infty} \frac{\beta\gamma^{i+1}\lambda^r(-1)^{i+k}(2i+j+1)!\xi!}{2^i i! j! k! (2i+2)! (\xi-k-1)! (\beta(k+1)-r)} \Bigg|_{\xi=\alpha(2i+j+2)} ; \quad \forall \quad r = 1, 2, \dots \quad (17)$$

The moment generating function, $M_X(t)$ of *REOGP* distribution is given by:

$$M_X(t) = E(e^{tx}) = \sum_{i,j,k,r=0}^{\infty} \frac{\beta\gamma^{i+1}(\lambda t)^r(-1)^{i+k}(2i+j+1)!\xi!}{2^i i! j! k! r! (2i+2)! (\xi-k-1)! (\beta(k+1)-r)} \Bigg|_{\xi=\alpha(2i+j+2)} \quad (18)$$

The quantile function, $Q_{X_{REOGP}}(q)$ of *REOGP* distribution is given by:

$$Q_{X_{REOGP}}(q) = \lambda \left\{ 1 - \left(\frac{\left[2 \log_e \left(\frac{1}{1-q} \right) \right]^{\frac{1}{2}}}{\gamma^{\frac{1}{2}} + \left[2 \log_e \left(\frac{1}{1-q} \right) \right]^{\frac{1}{2}}} \right)^{\frac{1}{\alpha}} \right\}^{-\frac{1}{\beta}} \quad 0 < q < 1 \quad (19)$$

Therefore, by setting $q = \frac{1}{4}$, $q = \frac{1}{2}$ and $q = \frac{3}{4}$; one obtains the 1st quartile, median and 3rd quartile of *REOGP* distribution respectively given as:

$$Q_{X_{REOGP}}\left(\frac{1}{4}\right) = \lambda \left\{ 1 - \left(\frac{0.7585}{\gamma^{\frac{1}{2}} + 0.7585} \right)^{\frac{1}{\alpha}} \right\}^{-\frac{1}{\beta}},$$

$$Q_{X_{REOGP}}\left(\frac{1}{2}\right) = \lambda \left\{ 1 - \left(\frac{1.7774}{\gamma^{\frac{1}{2}} + 1.7774} \right)^{\frac{1}{\alpha}} \right\}^{-\frac{1}{\beta}}$$

and

$$Q_{X_{REOGP}}\left(\frac{3}{4}\right) = \lambda \left\{ 1 - \left(\frac{1.6651}{\gamma^{\frac{1}{2}} + 1.6651} \right)^{\frac{1}{\alpha}} \right\}^{-\frac{1}{\beta}}.$$

The Shannon entropy, $\eta_{X_{REOGP}}$ of *REOGP* distribution is given by:

$$\begin{aligned} \eta_{X_{REOGP}} = & 2 + \frac{1}{\beta} + \log_e(\lambda) - \log_e(\alpha\beta) - \frac{1}{2\alpha} \log_e(\gamma) - \left(\frac{2\alpha - 1}{2\alpha} \right) \\ & (0.6931 - \zeta) - \left(\frac{\alpha + 1}{2\alpha\gamma} \right) \frac{\pi \operatorname{erfi}\left(\frac{\gamma}{2}\right) - Ei\left(\frac{\gamma}{2}\right)}{e^{\frac{\gamma}{2}}} \end{aligned} \quad (20)$$

where,

$$Euler's constant = \zeta = 0.577215664 \dots,$$

$$Exponential integral of x = Ei(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^x \frac{e^{-t}}{t} dt$$

and

$$imaginary error function of x = \operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The PDF of the i^{th} order statistic $X_{(i:n)}$ of *REOGP* distribution is given by:

$$f_{i:n}(x; \alpha, \beta, \gamma, \lambda) = \sum_{j=0}^{(n-i)} \sum_{k=0}^{(i+j-1)} \sum_{m,r,t=0}^{\infty} \tau_{jkmrt} x^{-(\beta t + \beta + 1)} \quad (21)$$

Hence, the PDF of the 1^{st} (minimum) order statistic $X_{(1:n)}$ of *REOGP* distribution is given by:

$$f_{1:n}(x; \alpha, \beta, \gamma, \lambda) = \sum_{j=0}^{(n-1)} \sum_{k=0}^j \sum_{m,r,t=0}^{\infty} \varepsilon_{jkmrt} x^{-(\beta t + \beta + 1)} \quad (22)$$

The PDF of the n^{th} (maximum) order statistic $X_{(n:n)}$ of *REOGP* distribution is given by:

$$f_{n:n}(x; \alpha, \beta, \gamma, \lambda) = \sum_{k=0}^{(n-1)} \sum_{m,r,t=0}^{\infty} \sigma_{kmrt} x^{-(\beta t + \beta + 1)} \quad (23)$$

where,

$$\tau_{jkmrt} = \frac{n!(-1)^{j+k+m+t} \beta \lambda^{\beta(t+1)} \gamma^{m+1} (i+j-1)! (k+1)^m (2m+r+3)(2m+r+1)! \xi!}{2^m (i-1)! (n-i-j)! (i+j-k-1)! (2m+2)! (\xi-t-1)! j! k! m! r! t!},$$

$$\varepsilon_{jkmrt} = \frac{(-1)^{j+k+m+t} n! \beta \lambda^{\beta(t+1)} \gamma^{m+1} (k+1)^m (2m+r+3)(2m+r+1)! \xi!}{2^m (n-j-1)! (j-k)! (2m+2)! (\xi-t-1)! k! m! r! t!},$$

$$\sigma_{kmrt} = \frac{n!(-1)^{k+m+t} \beta \lambda^{\beta(t+1)} \gamma^{m+1} (k+1)^m (2m+r+3)(2m+r+1)! \xi!}{2^m (n-k-1)! (2m+2)! (\xi-t-1)! k! m! r! t!}$$

and

$$\xi = \alpha (2m + r + 2).$$

2.4 Parameter Estimation and Asymptotic Confidence Bounds (ACB) for REOGP Distribution

The maximum likelihood estimates as well as the ACB of the parameters of REOGP distribution are derived and presented in this section.

2.4.1 MLE for REOGP Distribution

Given a random sample $X_1, X_2, \dots, X_n$ drawn from REOGP distribution; the likelihood function, $L(\alpha, \beta, \lambda, \gamma)$ is:

$$L(\alpha, \beta, \gamma, \lambda) = (\alpha \beta \gamma \lambda^\beta)^n \prod_{i=1}^n \left\{ x^{-(\beta+1)} \left(1 - \left(\frac{\lambda}{x} \right)^\beta \right)^{2\alpha-1} \left(1 - \left(1 - \left(\frac{\lambda}{x} \right)^\beta \right)^\alpha \right)^{-3} e^{-\frac{\gamma}{2} \left(\frac{\left(1 - \left(\frac{\lambda}{x} \right)^\beta \right)^\alpha}{\left(1 - \left(1 - \left(\frac{\lambda}{x} \right)^\beta \right)^\alpha} \right)^2} \right\} \quad (24)$$

and the corresponding log-likelihood function, $l(\alpha, \beta, \lambda, \gamma)$ is given by:

$$l(\alpha, \beta, \gamma, \lambda) = n \log_e (\alpha \beta \gamma) + n \beta \log_e \lambda - (\beta + 1) \sum_{i=1}^n \log_e x_i + (2\alpha - 1) \sum_{i=1}^n \log_e \left(1 - \lambda^\beta x_i^{-\beta} \right) - 3 \sum_{i=1}^n \log_e \left(1 - \left(1 - \lambda^\beta x_i^{-\beta} \right)^\alpha \right) - \frac{\gamma}{2} \sum_{i=1}^n \left(1 - \lambda^\beta x_i^{-\beta} \right)^{2\alpha} \left(1 - \left(1 - \lambda^\beta x_i^{-\beta} \right)^\alpha \right)^{-2} \quad (25)$$

By differentiating (25), partially, with respect to each of the four parameters, equating the resultant nonlinear system of equations to zero and then solving simultaneously; one obtains the ML estimates of the parameters that would be found to maximize the likelihood as well as the loglikelihood function as follows:

$$\begin{aligned} \frac{\partial l(\alpha, \beta, \gamma, \lambda)}{\partial \alpha} = 0 &\Rightarrow \frac{n}{\alpha} + 2 \sum_{i=1}^n \log_e \left(1 - \lambda^\beta x_i^{-\beta} \right) \\ &+ 3 \sum_{i=1}^n \left(1 - \lambda^\beta x_i^{-\beta} \right)^\alpha \left(1 - \left(1 - \lambda^\beta x_i^{-\beta} \right)^\alpha \right)^{-1} \log_e \left(1 - \lambda^\beta x_i^{-\beta} \right) \\ &- \gamma \sum_{i=1}^n \left(1 - \lambda^\beta x_i^{-\beta} \right)^{2\alpha} \left(1 - \left(1 - \lambda^\beta x_i^{-\beta} \right)^\alpha \right)^{-3} \log_e \left(1 - \lambda^\beta x_i^{-\beta} \right) = 0 \end{aligned} \quad (26)$$

$$\begin{aligned} \frac{\partial l(\alpha, \beta, \gamma, \lambda)}{\partial \beta} = 0 \Rightarrow & \frac{n}{\beta} + n \log_e \lambda - \sum_{i=1}^n \log_e x_i - (2\alpha - 1) \lambda^\beta \sum_{i=1}^n x_i^{-\beta} \left(1 - \lambda^\beta x_i^{-\beta}\right)^{-1} \log_e (\lambda/x_i) \\ & - 3\alpha \lambda^\beta \sum_{i=1}^n x_i^{-\beta} \left(1 - \lambda^\beta x_i^{-\beta}\right)^{\alpha-1} \left(1 - \left(1 - \lambda^\beta x_i^{-\beta}\right)^\alpha\right)^{-1} \log_e (\lambda/x_i) \\ & + \alpha \gamma \lambda^\beta \sum_{i=1}^n x_i^{-\beta} \left(1 - \lambda^\beta x_i^{-\beta}\right)^{2\alpha-1} \left(1 - \left(1 - \lambda^\beta x_i^{-\beta}\right)^\alpha\right)^{-3} \log_e (\lambda/x_i) = 0 \end{aligned} \quad (27)$$

$$\begin{aligned} \frac{\partial l(\alpha, \beta, \gamma, \lambda)}{\partial \lambda} = 0 \Rightarrow & \frac{n\beta}{\lambda} - (2\alpha - 1) \beta \lambda^{\beta-1} \sum_{i=1}^n x_i^{-\beta} \left(1 - \lambda^\beta x_i^{-\beta}\right)^{-1} \\ & - 3\alpha \beta \lambda^{\beta-1} \sum_{i=1}^n x_i^{-\beta} \left(1 - \lambda^\beta x_i^{-\beta}\right)^{\alpha-1} \left(1 - \left(1 - \lambda^\beta x_i^{-\beta}\right)^\alpha\right)^{-1} \\ & + \alpha \beta \gamma \lambda^{\beta-1} \sum_{i=1}^n x_i^{-\beta} \left(1 - \lambda^\beta x_i^{-\beta}\right)^{2\alpha-1} \left(1 - \left(1 - \lambda^\beta x_i^{-\beta}\right)^\alpha\right)^{-3} = 0 \end{aligned} \quad (28)$$

$$\frac{\partial l(\alpha, \beta, \gamma, \lambda)}{\partial \gamma} = 0 \Rightarrow \hat{\gamma} = \frac{2n}{\sum_{i=1}^n \left(1 - \lambda^\beta x_i^{-\beta}\right)^{2\alpha} \left(1 - \left(1 - \lambda^\beta x_i^{-\beta}\right)^\alpha\right)^{-2}} \quad (29)$$

It is obvious (26) through (29) are highly nonlinear in the parameters; hence, closed-form solutions do not exist and the corresponding estimates of the parameters can only be obtained using iterative numerical algorithms when using some given data sets.

2.4.2 ACB for REOGP Distribution

The ACB for the parameters of *REOGP* distribution can be found by estimating the elements of the variance-covariance (VCV) matrix whose elements constitute the inverse of the Fisher Information matrix whose elements are the negatives of the second derivatives of the log-likelihood function with respect to the unknown parameters. The VCV matrix is given by:

$$\begin{aligned} \Sigma(\omega) &= \begin{pmatrix} -\frac{\partial^2 l(\alpha, \beta, \lambda, \gamma)}{\partial \alpha^2} & -\frac{\partial^2 l(\alpha, \beta, \lambda, \gamma)}{\partial \alpha \partial \beta} & -\frac{\partial^2 l(\alpha, \beta, \lambda, \gamma)}{\partial \alpha \partial \lambda} & -\frac{\partial^2 l(\alpha, \beta, \lambda, \gamma)}{\partial \alpha \partial \gamma} \\ -\frac{\partial^2 l(\alpha, \beta, \lambda, \gamma)}{\partial \beta^2} & -\frac{\partial^2 l(\alpha, \beta, \lambda, \gamma)}{\partial \beta \partial \lambda} & -\frac{\partial^2 l(\alpha, \beta, \lambda, \gamma)}{\partial \beta \partial \gamma} \\ -\frac{\partial^2 l(\alpha, \beta, \lambda, \gamma)}{\partial \lambda^2} & -\frac{\partial^2 l(\alpha, \beta, \lambda, \gamma)}{\partial \lambda \partial \gamma} \\ -\frac{\partial^2 l(\alpha, \beta, \lambda, \gamma)}{\partial \gamma^2} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha}, \hat{\beta}) & Cov(\hat{\alpha}, \hat{\lambda}) & Cov(\hat{\alpha}, \hat{\gamma}) \\ & Var(\hat{\beta}) & Cov(\hat{\beta}, \hat{\lambda}) & Cov(\hat{\beta}, \hat{\gamma}) \\ & & Var(\hat{\lambda}) & Cov(\hat{\lambda}, \hat{\gamma}) \\ & & & Var(\hat{\gamma}) \end{pmatrix} \end{aligned}$$

The elements of the Fisher Information matrix can be obtained by negating the values of the Hessian matrix, which is the second derivative of (24) with respect to the parameters. Due to the complexity of the expressions, we omit the second

derivatives and make it available upon request.

Thus, in order to determine $(1 - \zeta)$ 100% approximate confidence intervals for the parameters, we use the following:

$$\hat{\alpha} \pm Z_{\frac{\zeta}{2}} \sqrt{\text{var}(\hat{\alpha})}; \quad \hat{\beta} \pm Z_{\frac{\zeta}{2}} \sqrt{\text{var}(\hat{\beta})}; \quad \hat{\lambda} \pm Z_{\frac{\zeta}{2}} \sqrt{\text{var}(\hat{\lambda})}; \quad \hat{\gamma} \pm Z_{\frac{\zeta}{2}} \sqrt{\text{var}(\hat{\gamma})}$$

Where $Z_{\frac{\zeta}{2}}$ is the upper $\left(\frac{\zeta}{2}\right)^{th}$ percentile of the standard normal distribution.

2.5 Monte Carlo Simulation Processes

This section is partitioned into two parts. The first is aimed at conducting Monte Carlo simulation to assess the potentials of *REOGP* distribution in fitting datasets from diverse field of human endeavor. The second part is aimed at showcasing the competitiveness of the distribution in fitting some real-life datasets in relation to other distributions reported in earlier studies.

To be able to generate a dataset (that would, most expectedly, come from *REOGP* distribution) based on simulation; the quantile function given in (19) was utilized to generate datasets with varying sample sizes, $n = 25, 50, 75, 100, 200$ and 400 for some selected parameters values: $\alpha = 0.01$, $\beta = 0.36$, $\gamma = 1.50$, $\lambda = 0.15$. The data-generating process was repeated 150 times (for each of the sample sizes) to obtain fairly representative estimates of the mean, bias, variance and mean squared error (MSE). The following table shows the result obtained (rounded to 6 dp):

Table 1 presents, among other things, the mean values of the estimates $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda})$ based on their corresponding actual parameter values $(\alpha = 0.01, \beta = 0.36, \gamma = 1.50, \lambda = 0.15)$ as the sample sizes increase gradually. It is obvious, with gradual increase in sample size, the mean values as well as biases of the estimates approach the actual (chosen) parameter values and zero respectively. In the same vein, the variances and MSEs of the estimates maintained a gradual decrease in magnitude with corresponding increase in sample size also.

3. Results and Discussion

In this section, a performance comparison with other distributions is undertaken to assess the capability of *REOGP* distribution. Two available data sets were utilized for that purpose. Data set I comprise a total of 63 data items relating to the strengths of 1.5cm glass fibers. Smith and Naylor (1987), Bourguignon *et al.* (2014), Ieren and Yahaya (2017), Ishaq and Abiodun (2020) as well as Yahaya *et al.* (2021) were among the few researches that made use of the dataset. Data set II is made up of 60 data items extracted from the webpage of Nigeria Centre for Disease Control (NCDC). The data set represents records of the number of daily (confirmed) cases of COVID-19 infections in Nigeria over a period of two months from 1st April to 30th May, 2020.

Table 1: Monte Carlo simulation results with parameters replicated 150 times at each experimental run

Simulation Number	Sample Size (n)	Estimates	Mean	Bias	Variance	MSE
1	25	$\hat{\alpha}$	0.057860	0.048860	0.000097	0.002483
		$\hat{\beta}$	0.351987	0.092987	0.000025	0.008671
		$\hat{\gamma}$	1.475036	-0.023964	0.000008	0.000582
		$\hat{\lambda}$	0.150000	0.001000	0.000000	0.000001
2	50	$\hat{\alpha}$	0.055186	0.046186	0.000065	0.002198
		$\hat{\beta}$	0.353340	0.094339	0.000017	0.008916
		$\hat{\gamma}$	1.475749	-0.023251	0.000005	0.000545
		$\hat{\lambda}$	0.150000	0.001000	0.000000	0.000001
3	75	$\hat{\alpha}$	0.053378	0.044378	0.000062	0.002031
		$\hat{\beta}$	0.354258	0.095258	0.000016	0.009090
		$\hat{\gamma}$	1.476254	-0.022746	0.000005	0.000522
		$\hat{\lambda}$	0.150000	0.001000	0.000000	0.000001
4	100	$\hat{\alpha}$	0.052104	0.043104	0.000038	0.001896
		$\hat{\beta}$	0.354902	0.095902	0.000010	0.009207
		$\hat{\gamma}$	1.476594	-0.022406	0.000003	0.000505
		$\hat{\lambda}$	0.150000	0.001000	0.000000	0.000001
5	200	$\hat{\alpha}$	0.050058	0.041058	0.000009	0.001695
		$\hat{\beta}$	0.355940	0.096940	0.000002	0.009400
		$\hat{\gamma}$	1.477154	-0.021846	0.000001	0.000478
		$\hat{\lambda}$	0.150000	0.001000	0.000000	0.000001
6	400	$\hat{\alpha}$	0.049489	0.040489	0.000000	0.001639
		$\hat{\beta}$	0.356229	0.097229	0.000000	0.009453
		$\hat{\gamma}$	1.477310	-0.021690	0.000000	0.000470
		$\hat{\lambda}$	0.150000	0.001000	0.000000	0.000001

Furthermore, this research made use of three R-packages, namely maxLik (Henningsen and Toomet, 2011), fitdistrplus (Delignette-Muller and Dutang, 2015) and AdequacyModel due to Marinho et al. (2019) to compare and assess the performance of the distribution with three other (extensions of Pareto) distributions, namely: Weibull-Pareto Distribution (WPD) due to Alzaatreh et al. (2013a); New Weibull-Pareto Distribution (NRPD) due to Nasiru and Luguterah (2015) as well as 4-parameter Odd Generalized Exponential-Pareto Distribution (OGEPPD) due to Yahaya and Abdullahi (2019). The proposed REOGP was assessed and compared with the aforementioned distributions based on some goodness-of-fit performance metrics namely, the Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Cramer-von-Mises statistic (W), Anderson-Darling statistic (A) and Kolmogorov-Smirnov (K-S) test statistic under which fall the D-statistic and p-value.

Data set I: The data set on the strengths of 1.5cm glass fibers is given as follows:

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Table 2 shows performance assessment result to determine the distribution that

offer most fit to the given data set among all the competing distributions having varying number of parameters.

Table 2: Performance comparison of *REOGP* distribution with three others based on dataset I

Model		REOGP	OGE PD	NWPD	WPD
Parameter Estimates	$\hat{\alpha}$	9.750826	3.945428	0.867646	9.447630
	$\hat{\beta}$	2.904986	2.735337	4.689657	0.552945
	$\hat{\lambda}$	1.196548E-01	0.000652	-	-
	$\hat{\gamma}$	5.010607E-05	0.084461	1.567679	0.263632
Log-likelihood		-15.272250	-22.158290	-17.196890	-18.956870
AIC		38.544490	52.316580	40.393780	43.913740
CAIC		39.234150	53.006230	40.800560	44.320520
BIC		47.117030	60.889110	46.823180	50.343150
HQIC		41.916110	55.688200	42.922500	46.442460
Cramer von Mises	W^*	0.237875	0.474317	0.283038	0.388763
Anderson Darling	A^*	1.306569	2.605237	1.550715	2.119677
K-S test	D	0.154860	0.142360	0.198680	0.190210
	p -value	0.097430	0.155500	0.013840	0.020950

Table 2 shows among other things the estimates of the parameters of *REOGP* whose values are $\hat{\alpha} = 9.750826$, $\hat{\beta} = 2.904986$, $\hat{\lambda} = 1.196548\text{E-}01$ and $\hat{\gamma} = 5.010607\text{E-}05$. It can easily be observed that, the proposed distribution (*REOGP*) fits the dataset better than all the other competitors, namely OGE PD, NWPD and WPD. It recorded the largest value of the loglikelihood (-15.272250) as well as smallest values of AIC (38.544490), CAIC (39.234150), BIC (47.117030) and HQIC (41.916110). Hence, *REOGP* should be preferred over all other competing distributions when fitting the given data set.

Figure 5 compares the performance of *REOGP* in relation to other competitors based on empirical and theoretical plots of PDFs and CDFs as well as Q-Q and P-P plots. The figure, further reveals, pictorially, how *REOGP* distribution fits the data set appropriately. It can be seen thereupon that; theoretical PDF and CDF plots agree appropriately with their empirical counterparts. Furthermore, the Q-Q and the P-P plots show an acceptable fit of *REOGP* upon the data set at both tails and center of the distribution. To further support the claim of better fit for *REOGP* on the data set over its competitors; the total time test (TTT) curve (Figure 6) being considerably concave indicates that, the data set has an increasing hazard rate function; and, models (such as *REOGP*) with increasing hazard failure rates (see Figure 4 for details) fits data sets, such as data set 1 appropriately.

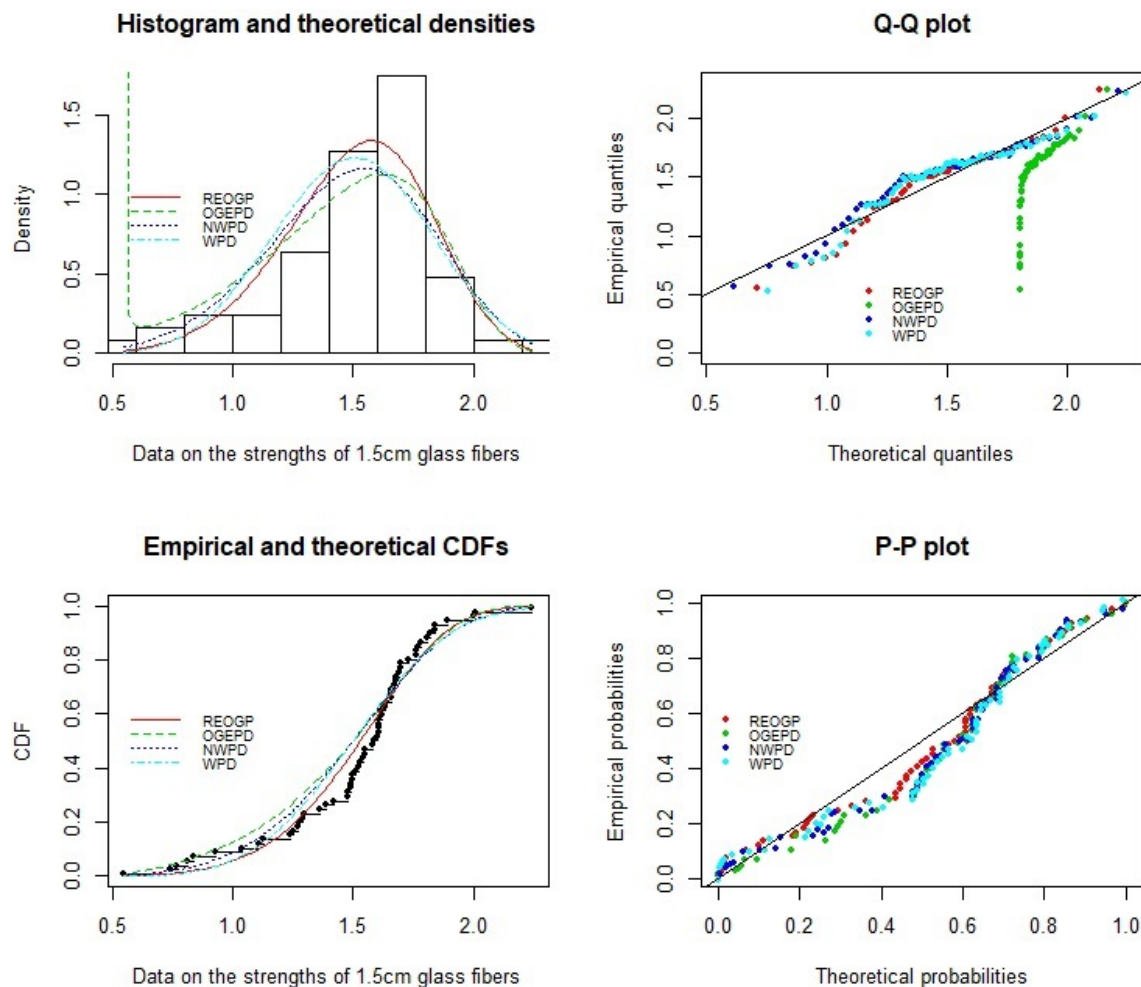


Figure 5: Empirical and Theoretical PDFs, CDFs, Q-Q & P-P plots showcasing fit of REOGP and others to dataset I

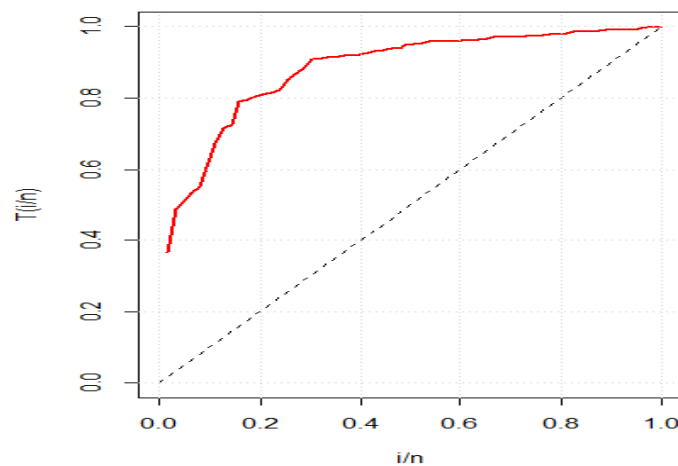


Figure 6: TTT curve plot for the glass fiber dataset

Figures 7, 8, 9 and 10 represent plots of the profiles loglikelihood against estimates of each of the *REOGP* distribution's parameters ($\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$ & $\hat{\gamma}$). It is obvious from these figures that, the likelihood function indeed attains its maximum value (-15.272250) at the reported estimated values of the parameters ($\hat{\alpha} = 9.750826$, $\hat{\beta} = 2.904986$, $\hat{\lambda} = 1.196548\text{E-}01$ and $\hat{\gamma} = 5.010607\text{E-}05$).

We now present, based on the results obtained, the Variance-Covariance (VCV)

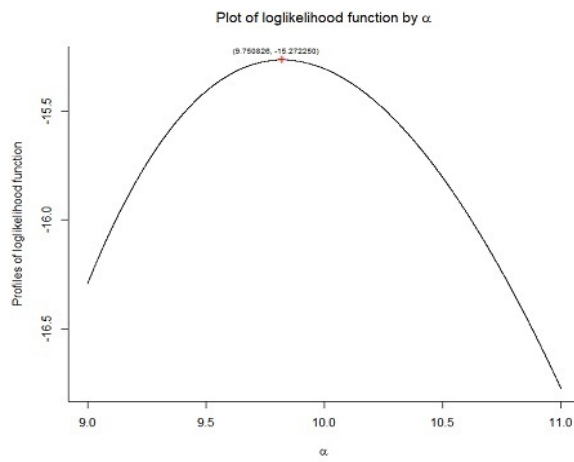


Figure 7: Plot of profile of log-likelihood by α

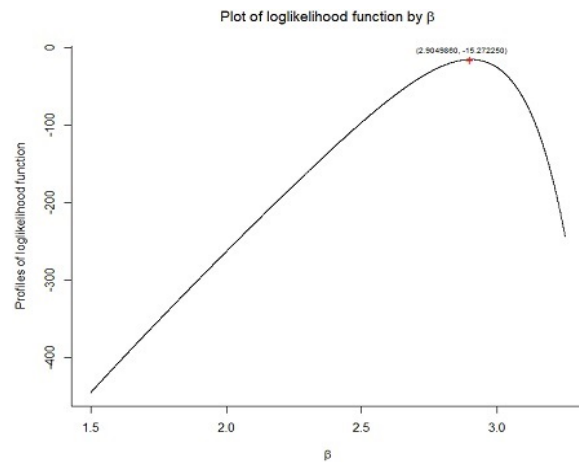


Figure 8: Plot of profile of log-likelihood by β

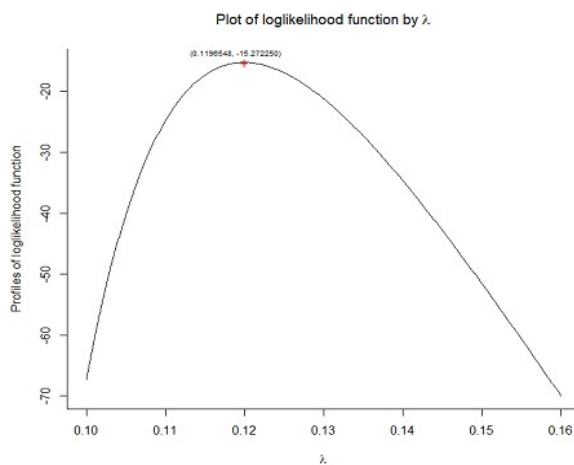


Figure 9: Plot of profile of log-likelihood by λ

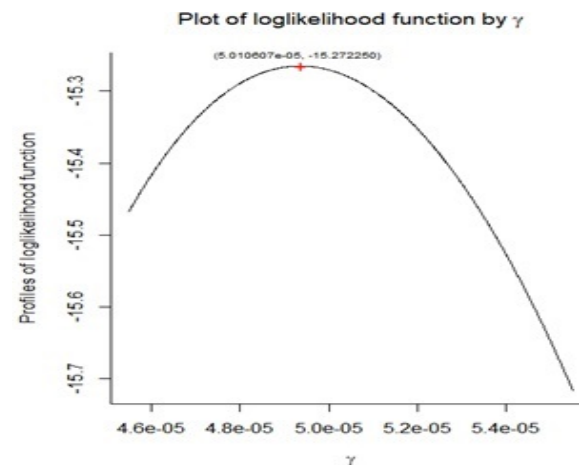


Figure 10: Plot of profile of log-likelihood by γ

matrix as:

$$\Sigma(\alpha, \beta, \lambda, \gamma) = \begin{pmatrix} 0.21998 & -0.06484 & -0.00313 & -2.17550 \times 10^{-11} \\ -0.06484 & 0.06919 & 0.00312 & 2.01493 \times 10^{-11} \\ -0.00313 & 0.00312 & 0.00014 & 2.36988 \times 10^{-12} \\ -2.17550 \times 10^{-11} & 2.01493 \times 10^{-11} & 2.36988 \times 10^{-12} & 4.50552 \times 10^{-18} \end{pmatrix}$$

Data set II: The data set of daily confirmed cases of COVID-19 patients in Nigeria from 1st April-30th May, 2020: 35, 10, 25, 5, 18, 6, 16, 22, 12, 17, 13, 5, 20, 30, 34, 35, 51, 48, 86, 38, 117, 91, 108, 114, 87, 91, 64, 195, 196, 204, 238, 220, 170, 245, 148, 195, 381, 386, 239, 248, 242, 146, 184, 193, 288, 176, 338, 216, 226, 284, 339, 245, 265, 313, 229, 276, 389, 182, 387, 553.

Table 3: Performance comparison of *REOGP* distribution with three others based on data set II

Model		REOGP	OGEPD	NWPD	WPD
Parameter	$\hat{\alpha}$	0.9808403	0.9445283	0.1545805	9.99997076
Estimates	$\hat{\beta}$	0.5536642	0.4389821	1.2698512	0.10085208
	$\hat{\lambda}$	5.743036e-05	0.1046263	-	-
	$\hat{\gamma}$	1.331994e-07	0.1734520	35.8105841	0.00783909
Log-likelihood		-364.9019	-402.1752	-366.4802	-366.5973
AIC		737.8038	812.3503	738.9604	739.1945
CAIC		738.5311	813.0776	739.3890	739.6231
BIC		746.1812	820.7277	745.2435	745.4776
HQIC		741.0806	815.6272	741.4181	741.6522
Cramer von Mises	W^*	0.2569977	0.3664516	0.2313456	0.3218038
Anderson Darling	A^*	1.4076680	2.004182	1.271442	1.755449
K-S test	D	0.1537959	0.452490	0.18947	0.17425
	p -value	0.117017	4.273e-11	0.02692	0.05232

From Table 3, it can be observed that, the values of the REOGP parameter estimates that provide the best fit for the given data set when compared with other distributions are $\hat{\alpha} = 0.9808403$, $\hat{\beta} = 0.5536642$, $\hat{\lambda} = 5.743036e - 05$, and $\hat{\gamma} = 1.331994e - 07$. It recorded the largest value of likelihood (-364.9019) as well as the smallest values of AIC (737.8038), CAIC (738.5311), BIC (746.1812), HQIC (741.0806) and the K-S (D) statistic value (0.1537959); the corresponding p -value (0.117017) is the largest. Hence, REOGP outperformed all other distributions based on data set II also.

Figure 12 shows the empirical and theoretical plots of PDFs and CDFs as well as Q-Q and P-P plots for data set II. From the figure it is evident that the proposed REOGP distribution fits the data set very well, in the sense that both theoretical PDF and CDF plots seem to be closer in mimicking their empirical counterparts. In the same vein, Q-Q and the P-P plots indicate that the proposed distribution paired well relative to other distributions in fitting the data set at both tails and the center of the distribution. The TTT curve of the data set (Figure 12) appears to be both convex and concave; signifying that, the data set has a U-shaped failure rate function; thereby making the proposed REOGP distribution an appropriate candidate for fitting this type of data sets (see Figure 4 for further details).

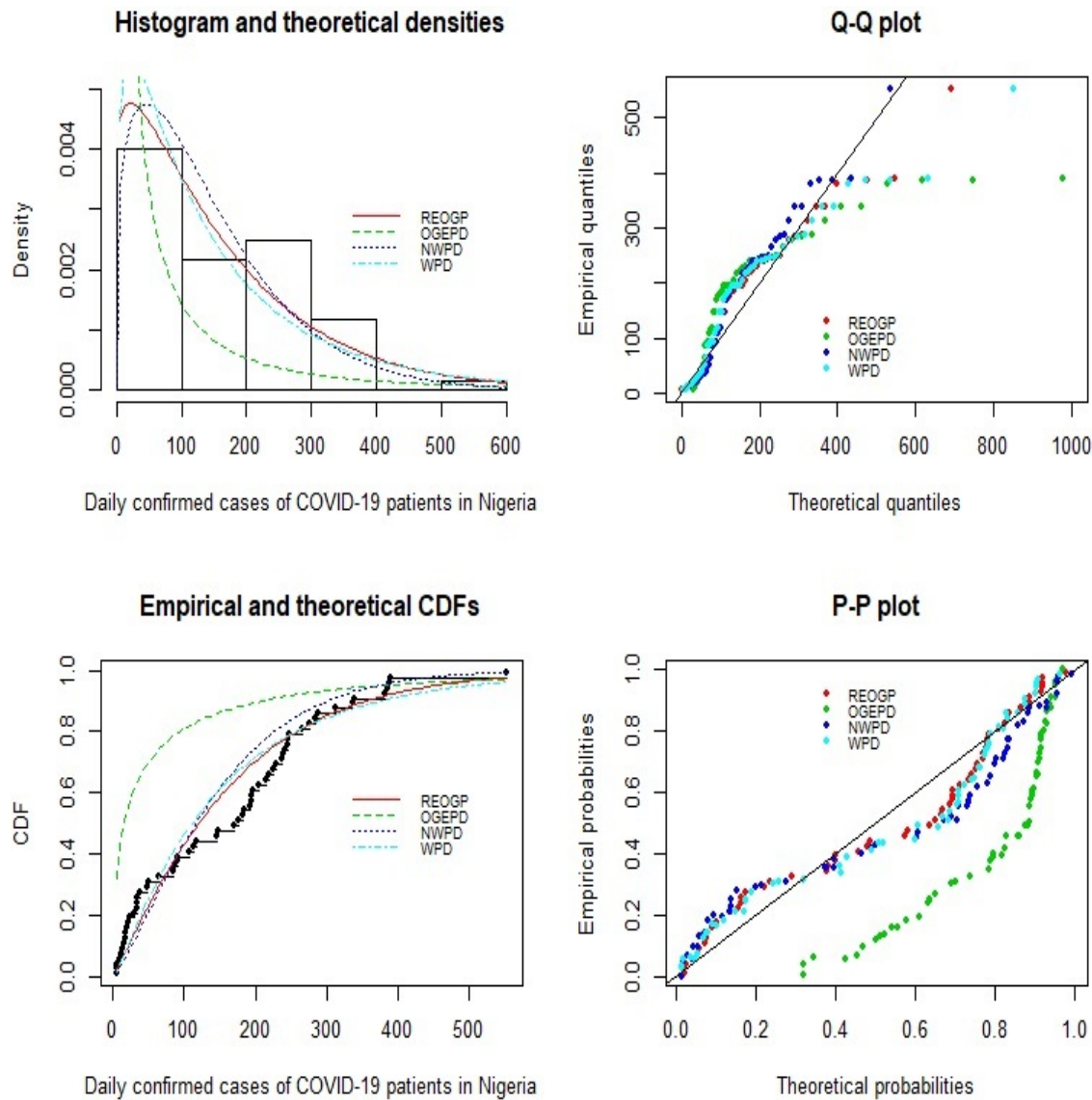


Figure 11: Empirical and Theoretical PDFs, CDFs, Q-Q & P-P plots showcasing fit of REOGP and others to data set II

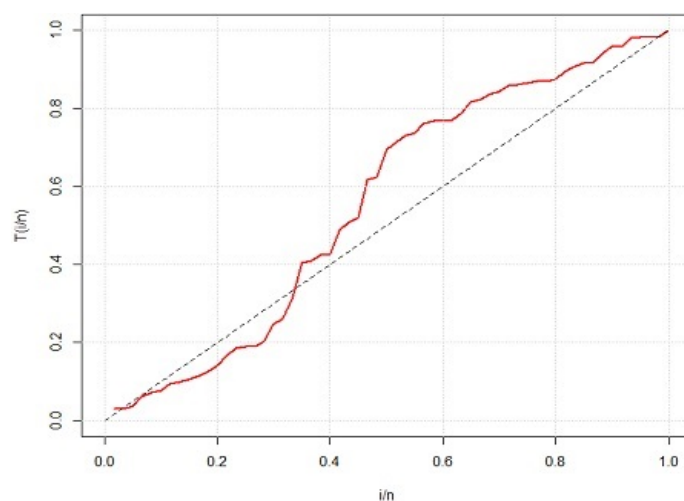


Figure 12: TTT curve for the data on COVID-19 confirmed cases in Nigeria

Figures 13, 14, 15 and 16 provide plots of profile of the likelihood function against those respective parameters of the distribution. It is evident from these

plots that, the estimated parameter values ($\hat{\alpha} = 0.9808403$, $\hat{\beta} = 0.5536642$, $\hat{\lambda} = 5.743036e-05$ and $\hat{\gamma} = 1.331994e-07$) constitute the approximate values of the unknown parameters ($\alpha, \beta, \lambda, \gamma$) that maximize the loglikelihood function in (24) for data set II.

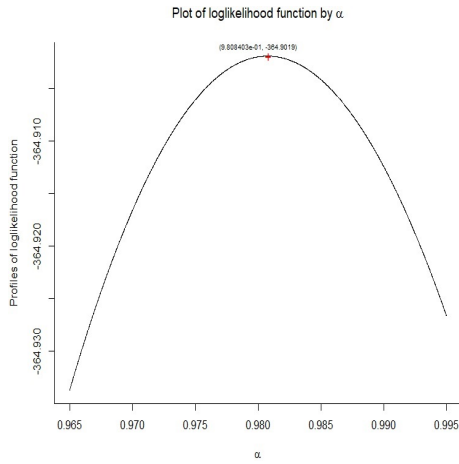


Figure 13: Plot of profile of log-likelihood by α

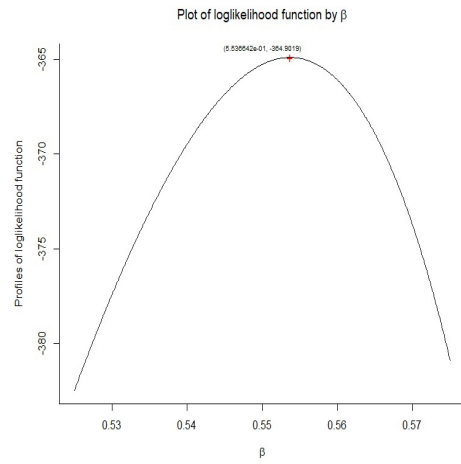


Figure 14: Plot of profile of log-likelihood by β

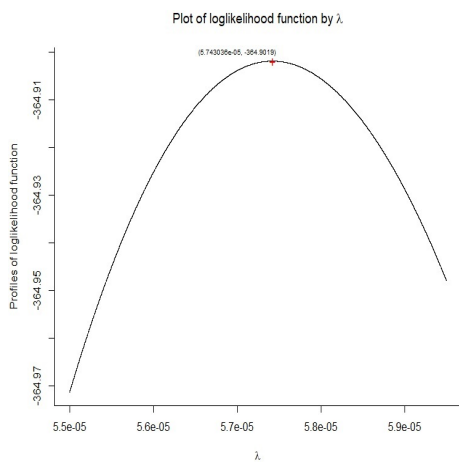


Figure 15: Plot of profile of log-likelihood by λ

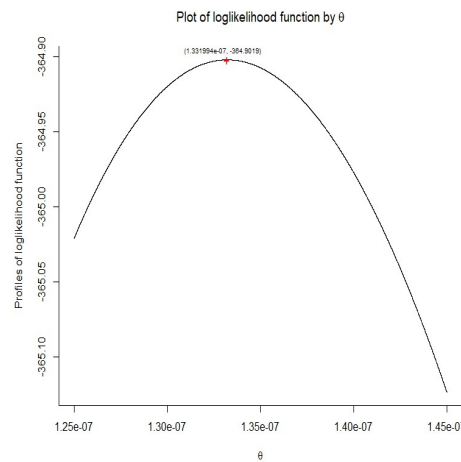


Figure 16: Plot of profile of log-likelihood by γ

The VCV matrix is given as:

$$\Sigma(\alpha, \beta, \lambda, \gamma) = \begin{pmatrix} 2.29762 \times 10^{-1} & 1.78419 \times 10^{-2} & 3.95249 \times 10^{-6} & -2.61964 \times 10^{-10} \\ 1.78410 \times 10^{-2} & 2.05581 \times 10^{-3} & 1.38434 \times 10^{-6} & -1.11969 \times 10^{-10} \\ 3.95249 \times 10^{-6} & 1.38434 \times 10^{-6} & 1.80071 \times 10^{-9} & -3.13944 \times 10^{-14} \\ -2.61964 \times 10^{-10} & -1.11969 \times 10^{-10} & -3.13944 \times 10^{-14} & 3.06114 \times 10^{-16} \end{pmatrix}$$

Hence, with ($\zeta = 5\%$) an approximate 95% confidence intervals for the unknown parameters ($\alpha, \beta, \gamma, \lambda$) as: ($0.03895 \leq \alpha \leq 1.92273$), ($0.46457 \leq \beta \leq 0.64276$), ($9.88195 \times 10^{-8} \leq \gamma \leq 1.67579 \times 10^{-7}$) and ($0 \leq \lambda \leq 0.00014$).

4. Conclusion

In this article, a variant of Rayleigh-Exponentiated Odd Generalized-X family of distributions proposed in earlier research is presented. The distribution is an extension of Pareto distribution with 2 additional parameters. The CDF, PDF, survival function, Hazard rate function as well as other mathematical properties comprising moments, moment generating function, quantile function, entropy as well as functions of order statistics were provided. The four parameters of the distribution estimated using MLE and the ACB of the parameters were presented. A Monte Carlo simulation runs was conducted and two real data sets were used to compare the goodness-of-fit of the proposed distribution with other existing distributions in the literature. The application results show the degree of competitiveness of the new distribution in modeling problems from diverse field of human endeavor.

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